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The role of intuition in the solving of optimization problems

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Abstract: This article presents the partial results obtained in the first stage of the research which sought to answer the following questions: (a) What is the role of intuition in university students' solutions to optimization problems? (b) What is the role of rigor in university students' solutions to optimization problems? (c) How is the combination of intuition and rigor expressed in university students' solutions to optimization problems? (d) Is there really an optimizing intuition? In the first part we provide reasons that make it plausible to consider intuition as a vector (metaphorically speaking) with three components: idealization, generalization, and argumentation. In the second part we present the experimental design of the research and analyze the data to answer the questions previously asked. The experimental design does not allow us to falsify the hypothesis that some students have an optimizing intuition.

1. INTRODUCTION

In the area of mathematics education various studies have been conducted into the relationships between intuition and rigor (e.g. Tall, 2006; Malaspina, 2007, Sirotic and Zazkis, 2007). The aim of the study presented in this paper was to examine the role of intuition and rigor in the solving of optimization problems by university students. More specifically, we address the following set of questions: (a) What is the role of intuition in university students' solutions to optimization problems? (b) What is the role of rigor in university students' solutions to optimization problems? (c) How is the combination of intuition and rigor expressed in university students' solutions to optimization problems? To answer these questions we considered it necessary to specify what is understood by formalization and, above all, by intuition in the context of the solving of optimization problems. Consequently, another question posed was: Is there really an optimizing intuition?

Providing answers to these questions requires theoretical constructs that enable the integrated study of “intuition”, “rigor”, “problem” and “formalization” within the context of the solving of optimization problems. Here we use some of the constructs proposed by the ontosemiotic approach to cognition and mathematics instruction (Godino, Batanero, and Font, 2007), as well as others associated with the “cognitive science of mathematics” (Lakoff and Núñez, 2000).

The paper is structured as follows. Section 2 describes the theoretical framework and seeks to explain what is meant by optimizing intuition in this study. In section 3 an integrated view of “intuition”, “rigor”, “problem” and “formalization” is presented, while section 4 sets out the methodology used in the research. Section 5 presents the partial results obtained in the first stage of the research designed to answer the questions stated above. Finally, section 6 presents some final conclusions and reflections.

2. THEORETICAL FRAMEWORK

This paper describes a study into the role of intuition and rigor in the solving of optimization problems from two specific theoretical frameworks: the onto-semiotic approach (mainly) and the cognitive science of mathematics (to a lesser extent).

The idea of investigating intuition and rigor from a specific theoretical framework has been taken up in other studies. An example would be the research on the relationship between intuition and rigor conducted within the theoretical framework proposed by Tall et al. In a recent paper (Tall, 2006), as well as in previous studies, Tall describes the relationships between intuition and rigor and claims that ideas of proof begin in the embodied and symbolic worlds in which "warrants for truth" relate to what can be seen and what can be calculated. Tall describes intuition as a growing cognitive faculty based on previous constructions of knowledge. Within the same theoretical framework, Semandi (2008) proposes the notion of “deep intuition of a concept”, and he links it to the notion of concept image. For Semandi, deep intuition is one level in the development of the concept image.

The present research is not focused on intuition in general but, rather, on a specific form, namely optimizing intuition. One way of classifying types of intuition is to consider the mathematical content to which intuition is applied. In the literature on mathematical intuition one can find research papers on, among others: (1) numerical intuition (Linchevski and Williams, 1999; Raftopoulos, 2002); (2) geometric intuition (Piaget and Inhelder, 1963; Fujita, Jones and Yamamoto, 2004); (3) intuition of infinity (Fischbein, Tirosh, and Hess, 1979; Tsamir and Tirosh, 2006); and (4) intuition of probability and combinatorial concepts (Fischbein and Grossman, 1997; Fischbein and Schnarch, 1997). This classification of intuition strengthens the question about the existence of an “optimizing intuition”.

Let us now summarize the theoretical framework used in this study.

2.1 The onto-semiotic approach (OSA)

Figure 1 (Font and Contreras, 2008, p. 35) shows some of the theoretical notions contained in the onto-semiotic approach to mathematics cognition and instruction (Godino, Batanero and Roa, 2005; Godino, Contreras and Font, 2006; Godino, Batanero and Font, 2007; Font, Godino and D’Amore, 2007; Font, Godino and Contreras, 2008; Font and Contreras, 2008; Ramos and Font, 2008; Godino, Font, Wilhelmi and Castro, 2009; Font, Planas and Godino, 2010). Here mathematical activity plays a central role and is modeled in terms of systems of operative and discursive practices. From these practices the different types of related mathematical objects (language, arguments, concepts, propositions, procedures and problems) emerge building cognitive or epistemic configurations among them (see hexagon in Figure 1).

Problem-situations promote and contextualize the activity; languages (symbols, notations, graphics) represent the other entities and serve as tools for action; arguments justify the procedures and propositions that relate the concepts. Lastly, the objects that appear in mathematical practices and those which emerge from these practices depend on the “language game” (Wittgenstein, 1953) in which they participate, and might be considered from the five facets of dual dimensions (decagon in Figure 1): personal/institutional, unitary/systemic, expression/content, ostensive/non-ostensive and extensive/intensive. Both the dualities and objects can be analyzed from a process-product perspective, a kind of analysis that leads us to the processes shown in Figure 1. Instead of giving a general definition of process the OSA opts to select a list of processes that are considered important in mathematical activity (those of Figure 1), without claiming that this list includes all the processes implicit in mathematical activity; this is because, among other reasons, some of the most important of them (for example, the solving of problems or modeling) are more than just processes and should be considered as hyper- or mega-processes.

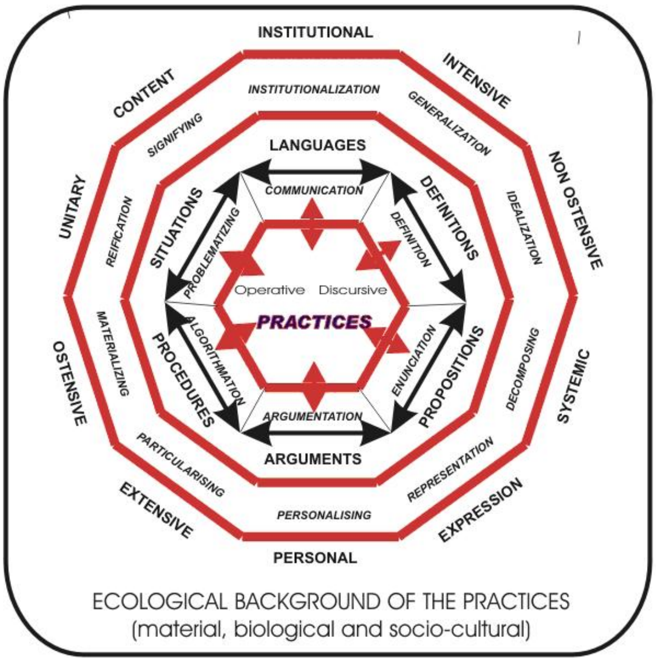


Figure 1. An onto-semiotic representation of mathematical knowledge

2.2 Cognitive science of mathematics

Lakoff and Núñez (2000) state that the mathematical structures people build have to be looked for in daily cognitive processes such as image schemas and metaphorical thinking. Such processes allow us to explain how the construction of mathematical objects is supported by the way in which our body interacts with objects in everyday life. To reach abstract thinking we need to use more basic schemas that are derived from the very immediate experience of our bodies. These basic schemas, called *image schemas*, are used to give meaning, through *metaphorical mappings*, to our experiences in abstract domains. Lakoff and Núñez (2000) claim that metaphors create a conceptual relationship between the source domain and the target domain, and they distinguish two types of conceptual metaphors in relation to mathematics: A) grounding metaphors, which relate a source domain outside mathematics with a target domain inside mathematics; and B) linking metaphors, which have both their source and target domains inside mathematics.

Lakoff and Núñez (2000) analyze four grounding metaphors whose target domain is arithmetic. In these four metaphors we can find an approximation to the relationship of order, which is vital to understanding the concepts of maximum and minimum. On the other hand, these authors also consider that the graphics of functions are structured through the metaphorical mapping of the "source-path-goal schema". Such a mapping conceptualizes the graphic of the function in terms of motion along a path, such as when a function is described as "going up", "reaching" a maximum, and "going down" again. In this way, the idea of the ups and downs of a road is essential to the understanding of the concepts of maximum and minimum.

2.3 The link between these frameworks

Several studies (Acevedo, 2008; Malaspina and Font, 2009) conducted within the OSA framework have developed theoretical connections between this approach and the theory of Lakoff and Núñez. Acevedo (2008) relates metaphorical processes to the sixteen processes shown in Figure 1, and does so by using the graphic representation of functions as the context for reflection.

In the papers by Acevedo (2008) and Malaspina and Font (2009) the epistemic/cognitive configuration (hexagon in Figure 1) construct is used to explain and specify the structure that is projected onto the linking metaphors. Here there is a source domain that has the structure of an epistemic/cognitive configuration (depending on whether the adopted point of view is institutional or personal) and which projects itself onto a target domain that also has the structure of an epistemic/cognitive configuration. This way of understanding the preservation of the metaphorical projection improves upon the explanation of such a preservation given by Lakoff and Núñez, who only provide a two-column table in which — mainly — properties and concepts are mixed (Lakoff and Núñez, 2000; Núñez, 2000).

The question that remains unresolved concerns what structure is projected in the case of a grounding metaphor. We believe that unlike linking metaphors, only some parts of the epistemic/cognitive configuration are projected. The specific study of each grounding metaphor will allow these parts to be identified.

2.4 Optimizing intuition in the present research

In this paper optimizing intuition refers to an a priori characterization of "optimizing intuition", and confirming the existence (or not) of this type of intuition is the aim of the experiment designed herein (see sections 4 and 5). To achieve this goal let us consider the following question as a context for reflection: Why are there people who consider it evident that a graphic which looks like a parabola that is shown to them (Figure 2(A)) has a maximum? To answer this question we use three of the processes considered in Figure 1 (idealization, generalization and argumentation), image schemas and metaphorical mappings in the way they were applied in Acevedo (2008) and Malaspina and Font (2009).

Above all, intuition has to do with the process of idealization (Font and Contreras 2008). Let us suppose that the teacher draws Figure 2(A) on the blackboard and that he talks about it as if he were displaying the graphic of a parabola, while simultaneously expecting that the students interpret the figure in a similar way. The teacher and students talk about Figure 2(A) as if it were a parabola. However, if we look carefully at Figure 2(A) we can see that it is not actually a parabola. Clearly, the teacher hopes that the students will go through the same idealization process with respect to Figure 2(A) and draw it on the sheet of paper as he has done. In other words, Figure 2(A) is an ideal figure (explicitly or implicitly) for the type of discourse the teacher and students produce about it. Figure 2(A), drawn on the sheet of paper, is concrete and ostensive (in the sense that it is drawn with ink and is observable by anyone who is in the classroom) and, as a result of the process of idealization, one has a non-ostensive object (the parabola) in the sense that one supposes it to be a mathematical object that cannot be presented directly. On the other hand, this non-ostensive object is particular. In the onto-semiotic approach this type of "individualized" object is called an extensive object.

Therefore, as a result of the process of idealization we have moved from an ostensive object, which was extensive, to a non-ostensive object that continues to be extensive.

The process of idealization is a process that duplicates entities, because in addition to the ostensive object that is present in the world of human material experiences, it gives existence (at least in a virtual way) to an idealized non-ostensive object. Font, Godino, Planas and Acevedo (2010) argue that the key notion of object metaphor is central to understanding how the teacher's discourse helps to develop the students' comprehension of non-ostensive mathematical objects as objects that have "existence". In fact, classical authors, such as Plato, considered intuition precisely as a bridge between the spatial-temporal world of ostensive objects and the ideal world of non-ostensive ones.

Intuition is also related to the generalization process because it can be considered as the process that allows us to see the general in the particular, a fact that is consistent with Fischbein's perspective (1987). In this case, there is also a generalization process according to which we consider this parabola to be a particular case of any curve that has "a similar shape to that of a parabola" (in the OSA this set is called an intensive object).

For classical authors such as Descartes (1986) the relationship between intuition and generalization, is necessary to explain one of the basic characteristics of mathematical reasoning: the use of generic elements. In his fifth meditation Descartes proposes that it is necessary to consider a specific object for intuition, one which cannot refer to itself but to particular objects, in order to be able to act. Intuition allows us to grasp what is general in what is particular (capturing the essence). At all events it is not the goal of the present paper to explore in depth the problem of the relationship between the generic element and intuition; rather we simply wish to emphasize that any characterization of intuition should consider its relationship with generalization. Additionally, when intuition is related to the generalization derived from the use of generic elements, the complementary and dynamic relationship between rigor and intuition is highlighted. That is, intuition can be found in the intermediate steps of a proof or problem solution.

When applied to geometric figures the combination of the idealization and generalization processes produces what Fischbein (1993) calls figural concepts. However, in the OSA the combination of these idealization and generalization processes, associated with the dualities ostensive/non-ostensive and intensive/extensive, are applied to any mathematical symbol, not just to geometric figures.

Given that intuition is usually considered as a clear and swift intellectual sensation of knowledge, of direct and immediate understanding, without using conscious and explicit logical reasoning, we can assume that in intuition there is no explicit argumentation even though there is an implicit inference. In the case shown in Figure 2 the inference could be, for instance, "as in the curve there is first a part that goes up and then a part that goes down, so there must be a point of maximum height".

The idea that intuition allows us to know the evident truth of specific mathematical propositions was a key element in the theory of mathematical truth, which remained valid until the appearance of non-Euclidian geometries. Briefly, that theory states that: 1) A statement is mathematically true if and only if the statement can be deduced from intuitive axioms; 2) Deduction from intuitive axioms is a necessary and sufficient condition of mathematical truth; 3) People perceive certain mathematical properties (e.g. axioms) as truths evident in themselves.

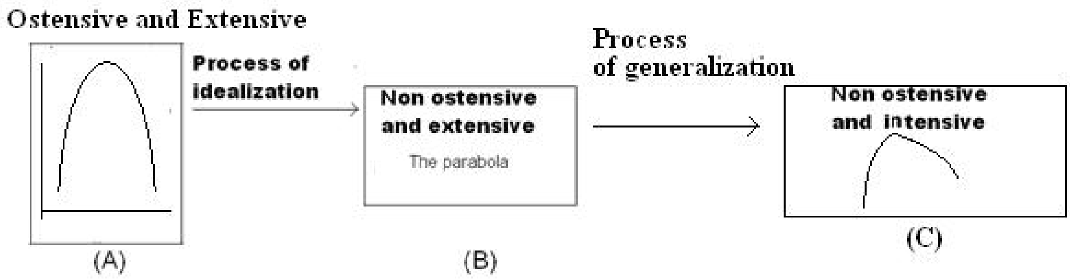


Figure 2. Idealization and generalization

We thus propose the use of a vectorial metaphor in which the intuitive process is a vector with three components (any of which could be "null" in some cases).

Intuition = (idealization, generalization, argumentation)

With this metaphor it can be seen that intuition acts upon universal mathematical ideas (which are present through their associated ostensive objects) to get to results that are considered true without (or almost without) an explicit argumentation. In fact, the different ways in which intuition can be understood differ in the emphasis they place on each of the three components of the "intuition vector".

This way of characterizing intuition draws on a series of common principles, or at least ones that do not contradict the characterization proposed by Fischbein (1987). Specifically, the properties of *self-evidence*, *intrinsic certainty* and *implicitness*, considered by Fischbein in his characterization of intuition, are related with the argumentation component of the intuition vector. The characteristic of *extrapolativeness* is linked to the idealization component, while the *theory status* is related to both the idealization and generalization components.

The task now is to find an explanation for the argumentation component in the specific case being dealt with here, i.e., how can we explain that it is evident that "because it first goes up and then comes down, there must be a maximum". We claim that there are reasons to assume that there is an *optimizing intuition* which leads this type of statement to be considered as evident.

In agreement with Fischbein we consider that experience is a fundamental factor in the development of intuition: "The basic source of intuitive cognitions is the experience accumulated by a person in relatively constant conditions" (Fischbein, 1987, p. 85). We also believe that the cognitive science of mathematics enables a more in-depth study of the mechanism through which experience generates optimizing intuition. Therefore, we will now use this cognitive approach to mathematics to describe the types of experience on which we consider optimizing intuition to be based and to explain how it is generated from them.

Optimizing intuition basically has its origin in two types of everyday experience. The first has to do with the fact that in everyday life we frequently have to face optimization problems, such as when we try to find the best way to go from one place to another (not necessarily the shortest), or when we try to make the best purchase, etc. This type of situation has an optimizing reasoning that seeks to find the best solution to a given situation.

The second type of experience is related to the fact that we are subjects who experience how certain physical characteristics, such as physical strength, health, etc., vary over time and pass through critical moments (maxima and minima). In this second type of experience we should also consider those related with the fact that we move frequently along roads which have ups and downs. We maintain that these two types of everyday situation allow us to make metaphorical mappings that contribute to the understanding of optimization problems. Below we explain a metaphorical mapping, of the grounding type, of what we call the domain of preferences.

Source Domain **Preference**	Target Domain **Optimization**
Toys (or another set)	Set A
Preference	Pre-order relationship
The most preferred	Maximum element
The least preferred	Minimum element

Table 1. Metaphorical mapping of preferences

In addition, bodily experience itself facilitates the appearance of the optimizing image schema, dependent from the source-path-goal schema (Malaspina and Font, 2009), which can subsequently be projected in more abstract domains. The metaphorical projection of the idea of the ups and downs of a path is essential to the understanding of the concepts of maximum and minimum.

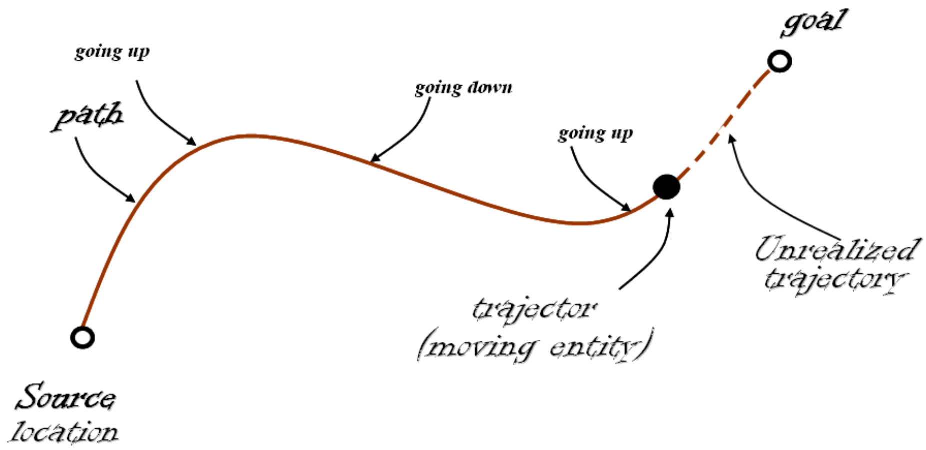


Figure 3. Optimizing image schema

We maintain that the metaphorical mapping of these domains of experience (preferences, consumer, etc.) and of the optimizing schema produces an intuitive understanding of optimization problems, and that it is this that allows us to answer intuitively the question with which we began this sub-section.

Using Fischbein's (1987) distinction between primary and secondary intuitions as a reference, we believe that this kind of intuition would be of the primary type, one which remains as stable acquisitions throughout life and which, due to the development of formal abilities, can gain in precision.

By considering that people generate an image schema that can be termed an "optimizing schema", we position ourselves, in accordance with Johnson (1987), at an intermediate point between two opposing perspectives regarding how information is stored in long-term memory. Some psychologists argue that long-term memory contains spatial images, whereas others refuse to accept that such mental images are stored in memory in a figurative format other than the propositional one. The proposal of Johnson (1987), whose origins can be found in Kant's theory of imagination, consists precisely in postulating certain schemas, known as image schemas, which are situated half-way between images and propositional schemas. Fischbein (1993), in his theory of figural concepts, also takes up an intermediate position in this controversy, although he limits himself to stating that the existence of figural concepts indicates that such an intermediate position is possible.

The OSA considers that when a student carries out and evaluates a mathematical practice it is necessary to activate a *cognitive configuration* (situation-problems, language, concepts, propositions, procedures and arguments). In terms of the cognitive configuration one of the characteristics of primary optimizing intuition is that the cognitive configurations of the solution to optimization problems solved with optimizing intuition do not explain why the solution obtained is the optimum, given that it is considered evident that the found value is the optimum. Further details on this matter can be found in Malaspina (2008).

In the OSA a practice carried out by a student which is consistent with a particular "belief" (correct or incorrect) can be considered to be the result of activating a "cognitive configuration". For example, if the subject's belief is that "if the first derivative of a derivable function at *a* is 0 then the function has a maximum or minimum at *a*", then this belief can be understood as a personal (cognitive) rule that will form part of the cognitive configuration which this subject will apply when carrying out mathematical practices in which the task is optimization of functions; it will hence be possible to observe regularly that when this student seeks maxima and minima of derivable functions, he will consider that they take place at every point in which the first derivative is 0 and he will not analyze the inflection points.

Stavy and Tirosh developed the theory of intuitive rules, which allows us to analyze students' inappropriate answers to a wide variety of mathematical and scientific tasks (Tirosh and Stavy, 1999; Tsamir, 2007). The theory claims that many common incorrect responses to mathematics and scientific tasks can be understood as evolving from a small number of intuitive rules, which are activated by specific external task features. From the perspective of the OSA the intuitive rules would be one of the elements of the cognitive configurations. In our view, explaining the practices of students according to a small number of intuitive rules could be improved by using the tool "cognitive configuration" (hexagon in Figure 1).

3. PROBLEM, RIGOR, FORMALIZATION AND INTUITION: AN INTEGRATED PERSPECTIVE

We consider that an *optimization problem* is one whose fundamental goal is to obtain the maximum or minimum value of a determined variable, keeping in mind the restrictions of the case, or one whose main goal is to obtain a strategy or a set of steps that constitute the best choice toward achieving a specific objective.

We believe that there is *formalization* if the participant uses equations, defines functions and applies theorems or mathematical results, if he uses graphics, diagrams, charts, or if he establishes a notation that allows a systematic handling of information or of the operations that he believes are necessary to perform. It is important to remember what Dubinsky (2000) says about formalism in mathematics.

> *By formalism, I am referring to sets of symbols, put together according to certain rules of syntax or organization, intended to represent mathematical objects and operations*. (p. 224)

As regards *rigor*, the study of this aspect is usually related to the proof, and thus, when the rigor of a solution to a problem is examined, we take this to be a proof or justification of the results (partial and final) obtained in such a solution. A rigorous solution to an optimization problem should show a good use of arguments, with logical sequences in its assertions and, specifically, with a justification that the result obtained is the optimum, using mathematical definitions, propositions and procedures.

In the present research we consider that a student displays rigor whenever he performs some type of proof that can be considered as an attempt to make some sort of justification: reasoning using an example, reasoning using a carefully selected example, reasoning using a generic example, logical reasoning using known propositions, complete induction, etc.

Since our goal was to study intuition within a problem-solving context, we focused mainly on the value of the "argument" component in the "intuition vector". Furthermore, because we were interested in non-trivial problems we focused particularly on intuitions in which immediacy (intrinsic evidence) and certainty (no need of proof) is present only in a few people (who are considered as "intuitive" people). However, other people, even when convinced that what they are conjecturing is true, are aware of the fact that it is necessary to make sure of this truth by providing arguments. Thus, we need to have the tools that enable us to deal comprehensively with the notions of "problem", "intuition", "formalization", and "rigor", which we have already mentioned. The constructs "epistemic configuration" and "cognitive configuration" proposed from the OSA (hexagon in Figure 1) served as the tools which made the study of this level of intuition operative.

A defining characteristic of an intuitive solution to a non-trivial problem is that there is practically no cognitive configuration, because the use of language is reduced to just what is necessary to give the correct answer. The properties, definitions and procedures are implicit. However, what usually occurs is that the block of the argumentation is not explicit or is restricted to referring to the evidence. Thus, the epistemic and cognitive configurations provide a comprehensive view of the notions of intuition, rigor, problem and formalization.

4. METHODOLOGY

Characterizing the intuitive solution to a non-trivial problem as if there were practically no cognitive configuration poses a methodological problem, because there are several possible explanations for correct answers that do not have the corresponding supporting argument. On the other hand, even rigorous and explicit proofs could be based on some intuitions. This problem led us to make certain decisions in the experimental design, specifically in the first stage of the research being conducted to answer the research questions.

The first decision was not to seek to justify the hypothesis regarding the existence of an optimizing intuition, but rather to design experimental situations that might enable such a hypothesis to be falsified.

The second decision was to exclude those optimization problems whose statements explicitly contain the graphic of the objective function, because in those cases the metaphoric projection of the “source-path-goal schema” and the optimizing scheme allows students to find the solution visually. The reason for excluding these problems is that this type of task does not offer a "natural need" for justification.

The third decision was to design an experimental situation to determine whether we should reject the hypothesis that there is an optimizing intuition and, more broadly, to examine what the role of intuition and rigor is in the solution of optimization problems by university students. In this experimental situation, we asked 38 students to solve optimization problems that were selected using controlled criteria.

Our first prediction was that if there was an optimizing intuition (of the primary type, as in the classification proposed by Fischbein), then a significant number of students facing non-trivial problems would provide intuitive solutions. These solutions would involve a written production that would allow the inference of cognitive configurations in which language would simply be that which was indispensable for giving the correct answer, while the properties, definitions and procedures would be implicit. The most characteristic scenario would be that the argumentation block is left as implicit or is restricted to referring to the evidence.

The existence of optimizing intuition is not the only possible explanation for the existence of correct answers without explicit justification (which could be due to a didactic contract that allows answers without justification). Hence, the students were selected in such a way that their didactic contract included the rule that solutions to problems should be justified.

To rule out the possibility that the existence of correct answers without justification could be caused by the didactic contract to which the students were used to, we made our second prediction: if the problem is solved by a group of students who are used to a didactic contract in which the results have to be justified, their written production would allow us to infer group cognitive configurations in which there would be an explicit argumentation of the answer. In other words, in the group solution of problems there would be very few solutions that could be characterized globally as intuitive (something which was expected in the individual answers) because the group would have applied the contract rule according to which solutions have to be justified.

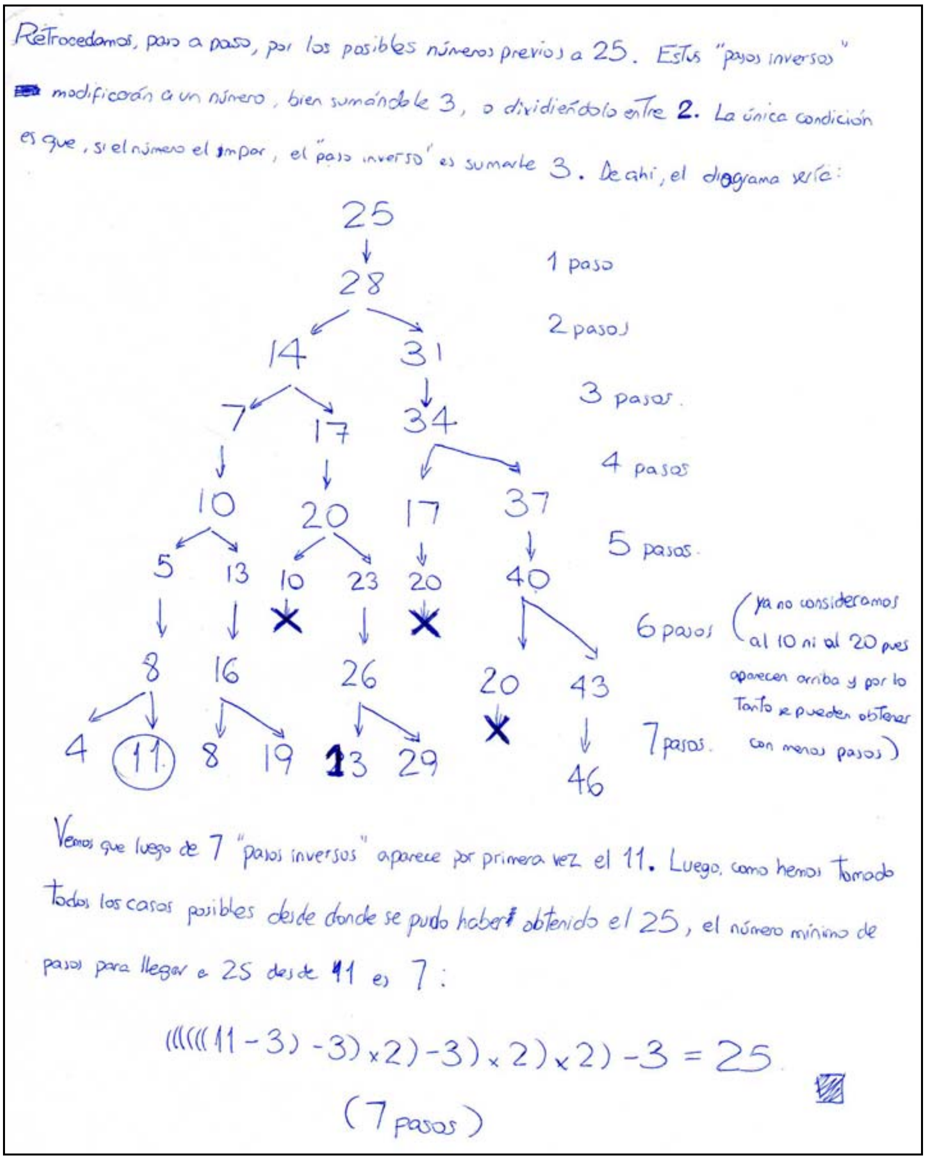
Our third prediction was that even in those cases in which the students’ cognitive configurations present explicit argumentations, some of the argumentation steps could also be an indication that there is an optimizing intuition. This type of intuition could even show up in some of the group responses.

Proposed problems, solutions, and epistemic configurations

In the empirical study we presented the students with the following problems. The first involves continuous variations and the second discrete variations.

> *Problem 1: On the Cartesian plane, find four points with integer coordinates so that they are the vertices of a parallelogram whose perimeter is 28 and whose area is the maximum.*
>
> *Problem 2: We say that a “step” is applied to a number if it is multiplied by 2 or diminished in 3 units. Find the smallest number of steps which should be applied to obtain 25, starting from 11.*

To identify solutions that could be used as a reference, teachers who are problem-solving experts analyzed the different possible solutions. For reasons of space we only show one of these here, that corresponding to problem 2, along with its corresponding epistemic configuration.



We go backwards, step by step, to the possible numbers before 25. These "inverse steps" will modify a number, either increasing it in 3 or dividing it by 2. The only condition is that if the number is odd, the "inverse step" is to add 3 to it. Thus the diagram would be like this:

Steps

6 steps (Here we do not consider numbers 10 and 20 because they appear above, and thus they can be obtained with fewer steps.**)**

We can see that the first time number 11 appears is after 7 "inverse steps". Then, because we have taken all the possible cases from which number 25 could be obtained, the minimum number of steps to get to 25 from 11 is 7.

Figure 4. "Expert" solution to problem 2

Mathematical objects	*Specifications*
Language	Step, multiply, decrease, divide, add. Representations: Tree diagram, numeric and operation symbols, =; ( )
Situation - Problem	Intra-mathematical problem, of arithmetic context, with discrete variations.

Concepts	Multiplication, subtraction, odd number, even number, order in the natural numbers. Inverse operations of multiplication and subtraction.
Procedures	▪ Initial estimations ▪ Construction of a tree diagram starting from 25, making the "inverse steps" of the defined: "divide by 2" and "add 3", and not developing branches which come from a number that has appeared in a previous level. ▪ Count the number of steps made to obtain 11.
Propositions	▪ It is impossible to get to 25 starting from 11 and only with multiplications by 2 or only with subtractions of 3 units. ▪ To get to an odd number, the last step cannot be a multiplication by 2. ▪ To get to 25, the step before the last must get to 28. ▪ Only one branch comes out of the odd numbers in the tree diagram that begins with 25. ▪ If we consider all the possibilities in the tree diagram that begins with 25, then the first time that 11 appears allows us to find the shortest path.
Arguments	▪ Thesis: The minimum number of steps to get 25 starting from 11 is 7. ▪ Argument: In a tree diagram which began with 25 and in which all possibilities have been considered, when number 11 is obtained for the first time, the number of steps to get to it is the minimum sought.

Table 2. Epistemic configuration of problem 2

It is important to note that in this problem the variable is the number of steps needed to get to 25 starting from 11.

Participants

The 38 students participating in the research were in the first or second year of their studies, in this case, different engineering majors. They were first asked to solve the problems individually and had to write down all their calculations, diagrams, drawings, etc. on a piece of paper they were given. They were then asked to solve the problems again but this time working in groups of no more than four students.

The first problem was designed so that the students did not restrict themselves only to giving intuitive answers, because this problem fulfills the three conditions that ensure their cognitive configurations have procedures, properties, definitions and explicit arguments. These three conditions are: 1) to be solvable using mathematical knowledge studied in previous courses; 2) to be proposed to students of a course in which it was important to provide well-supported solutions; and 3) to be a routine problem. The second problem was designed precisely to maintain the second condition and had the goal of facilitating the emergence of intuitive solutions. Because this problem was not a routine one and because the students did not have the specific epistemic rules, it was expected that the metarule of justifying solutions would be insufficient to ensure that their cognitive configurations have propositions, procedures, definitions and explicit arguments. Finally, the students completed a questionnaire so as to obtain information about their perceptions of the problems and their own solutions. Among others, one objective of this questionnaire was to confirm that the students' answers had the three abovementioned characteristics, a fact that was indeed confirmed.

5. PRELIMINARY ANSWERS TO THE RESEARCH QUESTIONS

In this section the individual and group solutions are analyzed and a preliminary answer is given to the research questions.

5.1 Analysis of the individual solutions

It is important to note that the theoretical framework for the analysis of solutions will fundamentally be made operational with the tools of the EOS. The approach of Lakoff and Núñez essentially gives us reasons to assume the existence of an optimizing intuition of the primary type as a consequence of life experiences.

Using the epistemic configurations mentioned in section 4 as a reference we have developed cognitive configurations of the students' answers. The reduction of the information led us to consider the following categories:

I. Cases in which only the answers were shown (a lack of explicit *arguments* and *procedures*). We examined the sub-cases of the correct answers and present the corresponding raw figures and percentages (Figure 5).

II. Cases in which formalizations were presented (use of formalized *language*). We examined the sub-cases of correct answers (IIa) and also, regardless of whether their answers were correct, the sub-cases in which they justified that the obtained result is the optimum (IIb) (use of *arguments*). The corresponding raw figures and percentages are presented (Figures 6 and 7).

III. Cases in which the students found what they were asked for. We examined the sub-cases of formalization (IIIa) (use of formalized *language*) and also, regardless of whether they had been formalized or not, the sub-cases where it is justified that the obtained result is the optimum (IIIb) (use of *arguments*) (Figures 8 and 9).

IV. Cases in which they attempted to justify that the obtained results are the optima (use of *arguments*). We examined the sub-cases of the correct explanation and present the corresponding raw figures and percentages (Figure 10).

V. Cases in which the presence of an optimizing intuition is observed in some part of the solutions presented by the students.

Due to the lack of space we will only show those solutions and cognitive configurations of students in cases I and II. More details can be found in Malaspina (2008). It should be remembered that problem 1 involves continuous variations (CV) and problem 2 discrete variations (DV).

Case 1

In the students' solutions we found the correct answer, but we did not find the procedures or explicit arguments. Neither could we identify which propositions they had used.

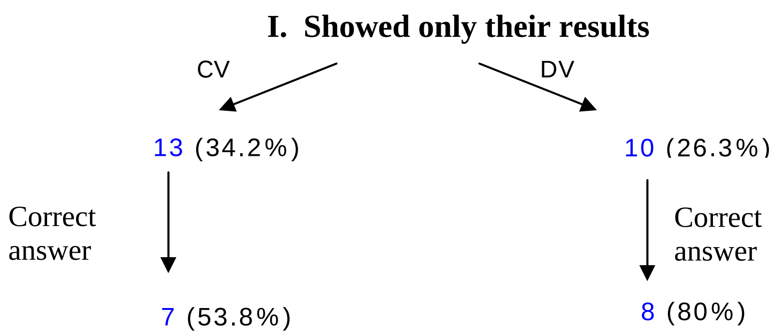


Figure 5. Analysis of the lack of arguments or procedures

It can be observed that in the discrete variation problem the percentage of students who showed only their results (10) is less than that in the continuous variation problem (13), but the percentage of these students who gave the correct answer (8 from 10) is greater than in the problem of continuous variation (7 from 13). Therefore, we could say that in the discrete variation problem there is a better intuitive approach than in the problem of continuous variation, or that the degree of effectiveness was greater when the students tried to solve the problem of discrete variation. As we had anticipated, the intuitive answer was given more often in the problem of discrete variation. It is important to note that in contrast with what was expected, there was a non-negligible number of intuitive correct responses in the CV problem.

Case II

As previously mentioned the formalization criterion is quite broad. Additionally, because we worked with students in their second or third semester (between 17 and 18 years old), we were not especially demanding, but we do distinguish those who only write some numbers or draw only a parallelogram from those who use algebraic expressions, equations, functional notation, theorems, diagrams, their own notation, etc.

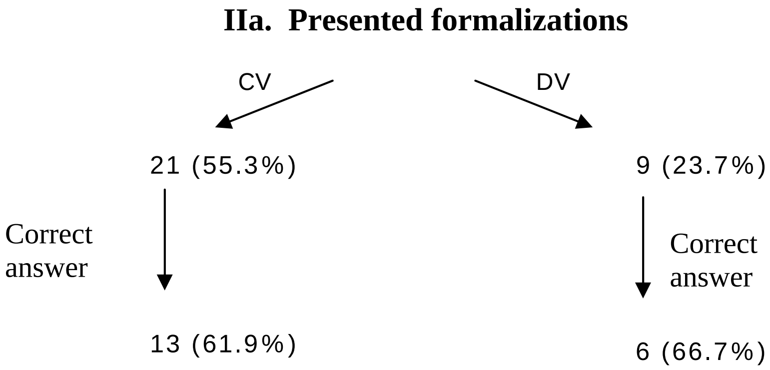


Figure 6. Analysis of the use of a formalized language

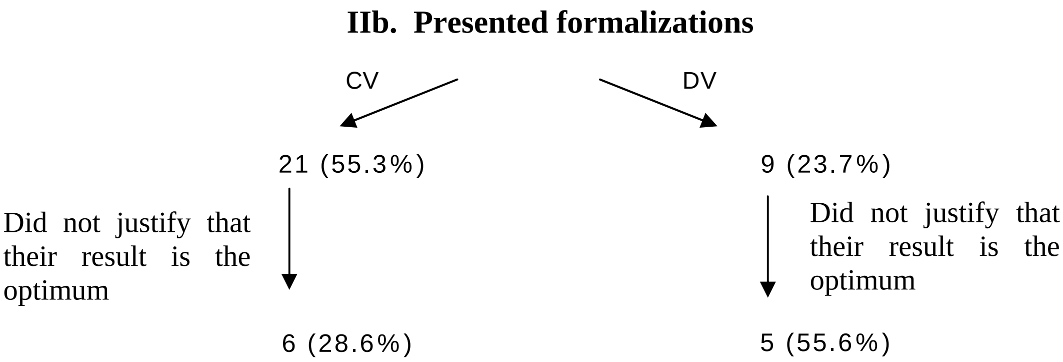


Figure 7. Analysis of the use of arguments

In the problem of discrete variation we expected to see a not very high percentage of cases in which there were formalizations. However, it is surprising to find a not very high percentage (55.3%) in the problem of continuous variation as well. One explanation that is consistent with the specific solutions found is the existence of deficiencies in the formal handling of arguments, procedures and propositions such as those described in the epistemic analysis of the problems, even though the percentage of those who get a correct answer using formalization is not very high.

We could say that quantitatively the deficiencies are more serious when the students solve the problem of discrete variation (Figure 6). However, it is important to highlight the solutions to this problem which use formalization, have the correct answer, and display a rough argumentation of the optimum nature of the solution found.

Another way to examine the cases that presented formalizations is to observe whether the students justified that the result they obtained is an optimum value, regardless of whether the answer itself is right or wrong (Figure 7). One solution to an optimization problem should include the justification that the result obtained is the optimum. However, a considerable percentage of students do not offer such a justification, even when they formalize, and this is especially so in the problem of discrete variation.

Case III

The results shown in Figures 8 and 9 indicate that there are formative deficiencies in the formalization and scientific attitude that might enable students to channel adequately the conjectures and intuitive estimates regarding the problem. We can see that not many of them found a correct answer using a formal language, especially in the problem of discrete variation, and there were even fewer who, having found the correct answer, were able to present an argument stating that the answer fulfilled the criterion of being the optimum answer asked for in the problem.

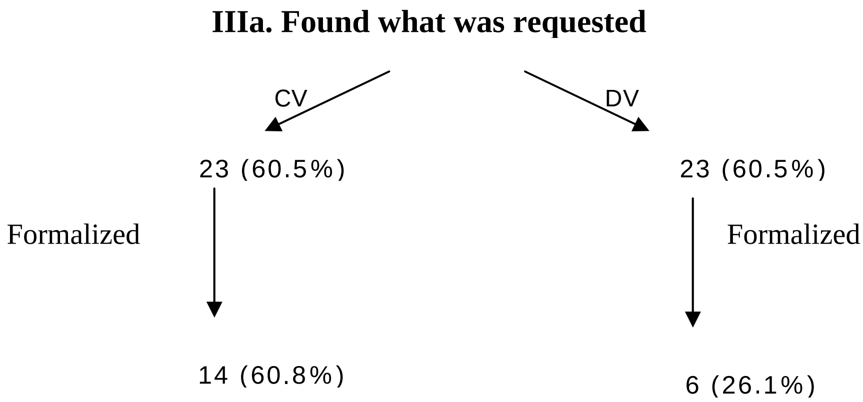


Figure 8. Analysis of the correct answer using a formalized language

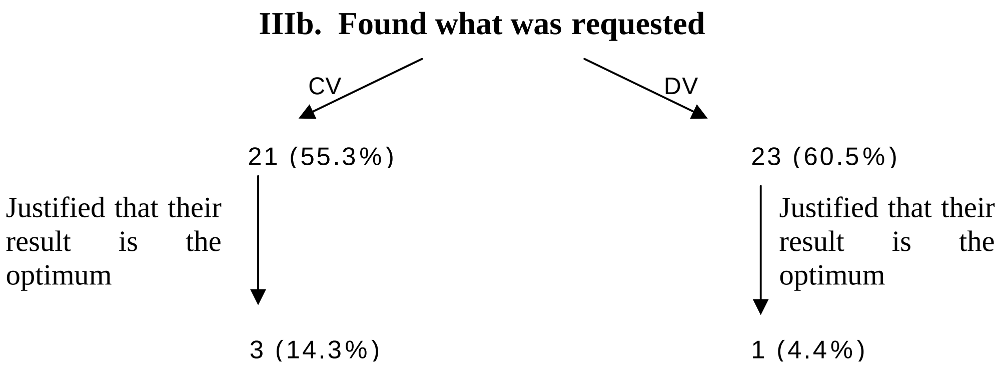


Figure 9. Analysis of the correct answer with an argument for the optimum result

Very few students found what was requested using formalization, justifying that what had been obtained was the optimum: 3 from 38 in the problem with continuous variation and 1 from 38 in that with discrete variation. Only one student (2.6%) found what was requested in this way in both problems.

Case IV

We can see that only a few students tried to justify that the results they had obtained were the optimum, and of these, even fewer gave a correct explanation.

IV. Tried to justify that their results were optimum

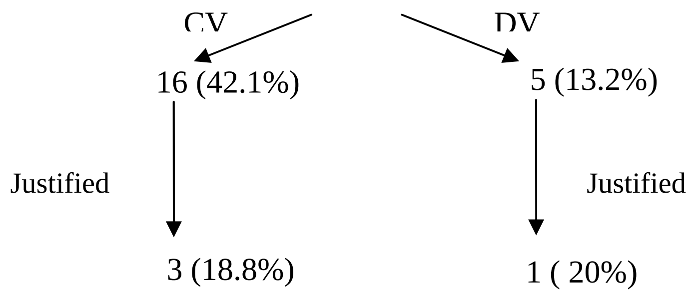


Figure 10. Analysis of the argumentation as to whether the results were optimum**.**

In the problem of discrete variation, many believed that it was enough to obtain a solution that seemed to be convincing. These are deficiencies in rigorous thinking, in the use of a formalized language, and in the ability to argue so as to prove the validity of results. However, we also found cases in which the use of what seems to be algebraic language and the search for formal justifications moves students away from the chance to take a more natural look at the given situation, especially when they had to solve the problem with discrete variation.

Case V

When we examined the solutions and cognitive configurations we found several assertions that did not have a justification but which were in the right direction as regards finding the answer. This could be considered as an indication that there is an optimizing intuition. Thus, in problem 1 we found 18 cases in which it is assumed or stated that the parallelogram being sought is a rectangle, or more specifically, a square or a parallelogram with equal sides, or that the numbers whose product is the maximum and whose sum is 14 must be both equal to 7.

Example of Case 1

Let us consider an example of a correct solution to the problem of continuous variation, placed in category I, and its corresponding cognitive configuration. Such a configuration reveals that when these students have to solve the problem, they only get to what our research has called an intuitive approximation. Because these students have already taken differential calculus and have a didactic contract that demands that they justify their answers, we can assert that they have not been significantly influenced by what they learned in their studies to go beyond what is an intuitive solution.

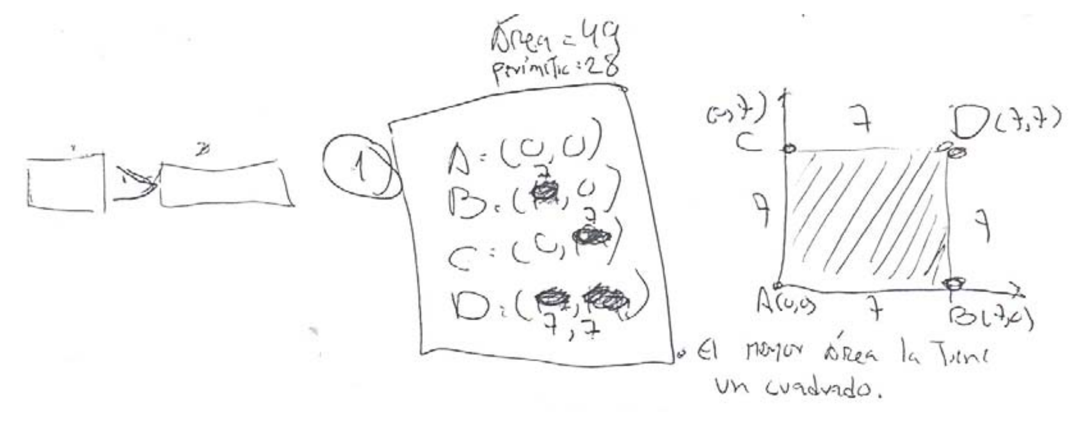


Figure 11. Student No. 29’s solution to problem 1

Language: Representations: Represent a square and a rectangle related with the greater-than symbol. Draw a square with a system of Cartesian coordinates. Symbolic expressions. Specify the coordinates of the vertices with letters and ordered pairs, corresponding to a square of side 7 and with a vertex in the origin.

Problem-situation: Rectangles isoperimetric problem.

Concepts: Parallelogram, square, perimeter, area, vertex, integer numbers. Coordinates of a point.

Propositions: (Implicit) A square is a parallelogram. A square whose perimeter is equal to that of a rectangle has a bigger area than the rectangle. A square whose perimeter is 28 has sides of length 7 and an area of 49.

Procedures: (Implicit) Calculation of the area of the square.

Arguments: None.

Table 3. Cognitive configuration for student No. 29’s solution to problem 1

It is clear that there is a lack of explicit procedures, propositions and arguments, and it naturally follows that this cognitive configuration is very different from the reference epistemic configuration.

If we look at the block of propositions of this student’s cognitive configuration (Table 3) we can see that there are three propositions that have been implicitly used. It is important to distinguish these properties on the basis of their mathematical relevance. That “A square whose perimeter is equal to that of a rectangle has a bigger area than that of the rectangle” is a property of isoperimetric quadrilaterals whose relevance is clearly greater than that of the other two (“A square is a parallelogram” and “A square of perimeter 28 has sides of length 7 and an area of 49”) While we do not have enough clues to assert that the first is known by the student, we can state that the other two, which are less relevant, are known by the students.

If this proposition is not known by the student, a plausible explanation is that we are dealing with an intuitive process. In that case, the student’s answer must contain some indicator of the three components of the intuition “vector”. It is plausible to assume that processes of idealization and materialization have occurred in the students’ solutions

(Figure 12), because the figures of the square and the rectangle on the left-hand side of the answer are the materialization of the mathematical objects "square" and "rectangle".

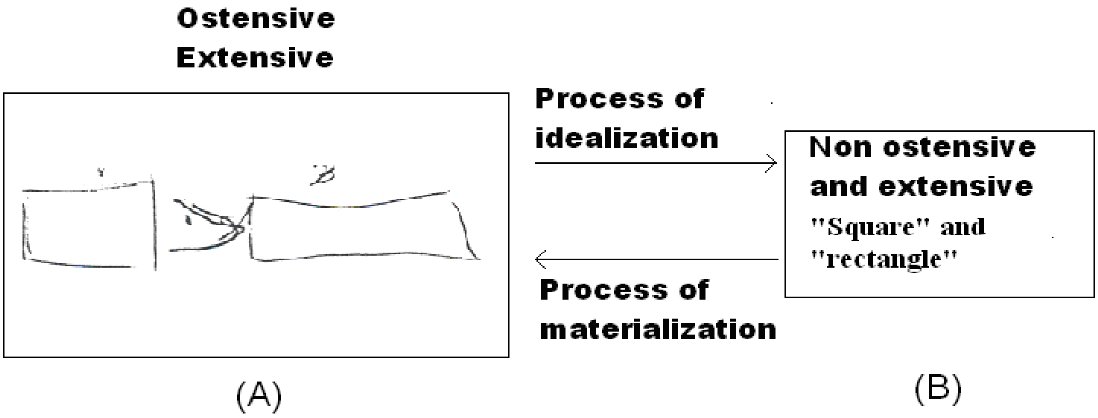


Figure 12. Processes of idealization and materialization

The student writes about Figure 12(A) as if it were one square and one rectangle. If we look carefully at Figure 12(A) we observe that: (1) the figure on the left looks more like a trapezium than a square; (2) in the two figures the lines are curved rather than straight like the segments of a line; (3) there are sides that are not connected, etc. Figure 12(A), which is drawn on the sheet of paper, is concrete and ostensive, and as a result of the process of idealization there are two non-ostensive objects (square and rectangle), in the sense that they are assumed to be mathematical objects that cannot be presented directly. On the other hand, these non-ostensive objects are particular. Therefore, as a result of the process of idealization we have moved from an ostensive object which was extensive to a non-ostensive object that continues to be extensive.

In this case, it can be argued that there is also a generalization process according to which the student considers that the relationship he is representing between the areas of this square and this rectangle (particulars), drawn in his answer, is valid for any isoperimetric square or rectangle. In fact, the figures of the square and the rectangle are handled as if they were generic elements. Finally, we can assume that the student believes that the validity of this relationship is evident, because he does not consider it necessary to provide any argumentation to justify it.

Example of Case II

In this problem we have considered the drawing of a tree diagram as the argumentation in which all possibilities can be inspected, even though the student did not argue explicitly that he had given the solution through a direct and exhaustive inspection of all possible cases.

We show here one of the few solutions to the discrete variation problem in which there is formalization and in which the student gives the correct answer; the corresponding cognitive configuration has a lot in common with the epistemic configuration that is used as a reference, which we have consequently omitted.

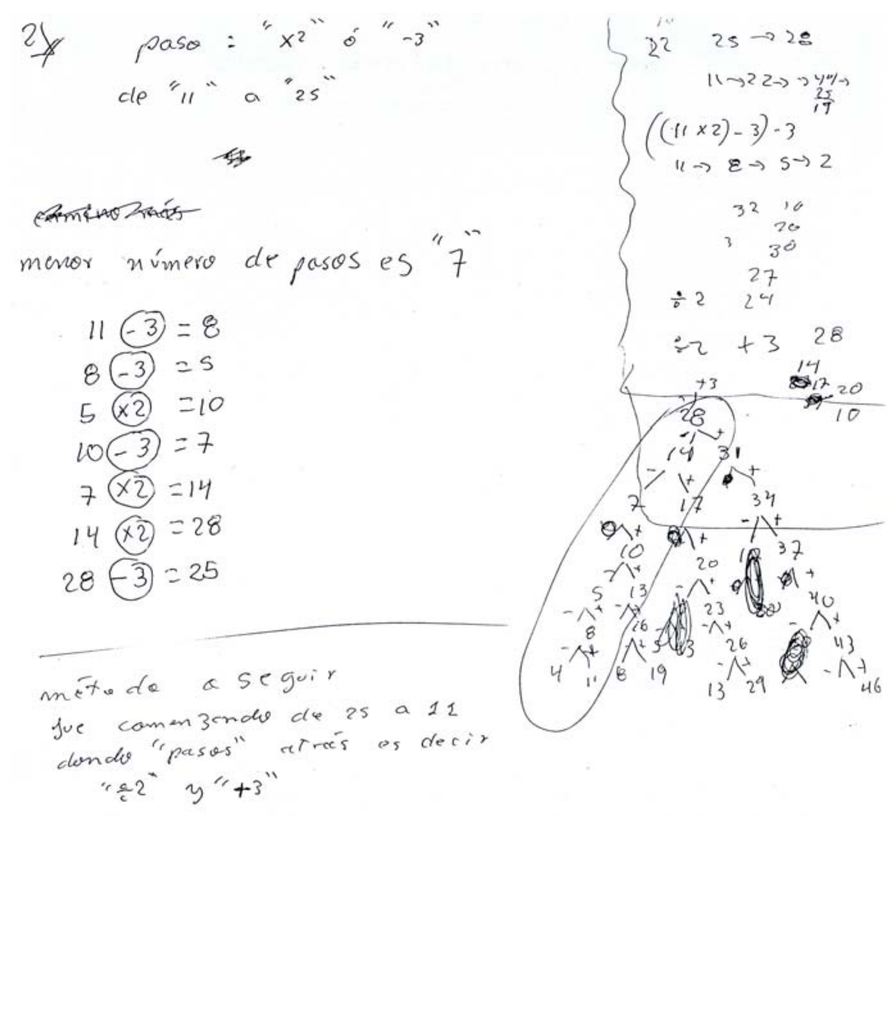
	Step smallest number of steps is "7" method followed: we started from 25 to 11, making "steps" backwards, that is "divided by 2" or "added 3"

Figure 13. Student 6's solution to problem 2.

Even though the answer to this problem, which uses a tree diagram to cover all the cases, has not been considered in our analysis as intuitive, it is important to note that this type of solution clearly shows the relationship between rigor and what could be called a finitist intuition. Once all possible cases have been constructed and when their number is finite, direct inspection gives us the certainty that the answer is correct.

5.2 Group solutions

The existence of an optimizing intuition is not the only explanation for the presence of correct answers without an explicit justification. To rule out the possibility that these answers could be caused by the type of didactic contract to which the students were used to, we made a second prediction in case the problem was solved by a group of students used to working under a contract in which they are expected to justify their results. The written production of these students would allow us to infer group cognitive configurations which contain an explicit argument to support the answer. In other words, in the group solutions there would be very few that could be globally characterized as intuitive, because the group would have used the rule "solutions have to be justified".

To see if the prediction was correct or not we asked the students to solve again the two problems they had previously solved individually, but this time in groups of no more than four members. We especially emphasized the actions targeted at making students fully aware of the rule that they had to justify their answers, and therefore they were literally asked: "Present group solutions to both problems, rigorously justifying the optimum values obtained."

The results obtained for problem 1 were as follows:

1. All groups found what was requested.
2. Nine groups used a formalized language and four did not use differential calculus. These nine groups tried to provide arguments to justify that the value they had obtained was the optimum.
3. Two groups correctly explained why the obtained value was the optimum, and they were among the ones who did not use differential calculus to solve the problem.
4. Five groups used differential calculus, but none justified correctly that the obtained value was the optimum.
5. Group 5 showed only its results and did not give any explanation about the maximum character of the obtained value.

We believe there are reasons to assert that optimizing intuition was also present in several group solutions and with a higher frequency than expected, especially in this case because the problem had the three characteristics mentioned at the end of section 4 and because the students were already used to a didactic contract that made them justify their assertions. Below we give some of these reasons:

a) None of the five groups which used differential calculus examined whether the value that equates to zero the first derivative of the function that expresses the area of the parallelogram is a maximum value. However, this is a task that should be done by any student who has been trained in mathematical rigor and who follows the didactic contract of studying differential calculus. We believe that one reason why some of these students did not justify their answers was that they used an optimizing intuition, which makes them see as evident and within the context of the problem that the value which makes the first derivative equal to zero is the maximum value.

b) The lack of argumentation in Group 5, even though they had been asked to justify their answer, suggests that these students used optimizing intuition, which made them consider that the solution they presented was the optimum.

The results obtained for problem 2 were:

1. All the groups found what was requested.
2. Nine groups used a formalized language, which was basically the use of some kind of symbols other than the mere expression of arithmetical operations.
3. Eight groups tried to justify that the result obtained was the optimum.
4. Three groups correctly explained why the obtained value was the optimum.
5. Groups 5 and 10 showed their results and did not offer any explanation about the minimum character of the value obtained.

We consider that there are reasons to believe there was optimizing intuition in several of the group solutions to this problem. Some of these reasons are:

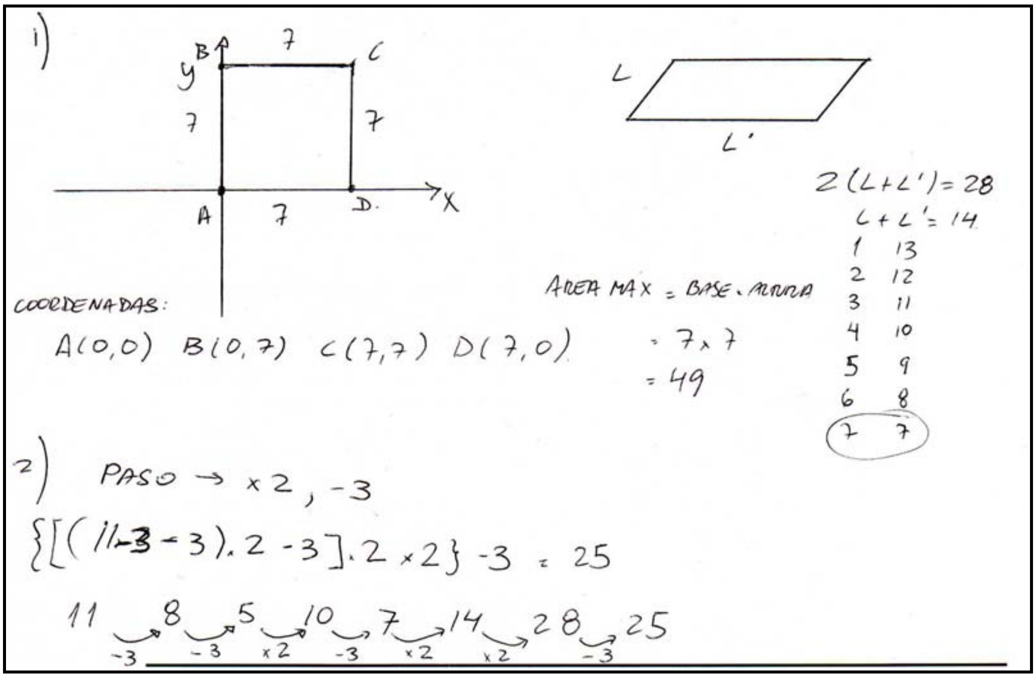
a) Groups 1 and 7 gave “intuitively true” reasons to support their procedures. They did not analyze the different possible cases.

b) Group 8 examined two cases with which it got to number 25 with ten and nine steps, respectively. It then showed the solution with seven steps, asserting that it was the optimum. We believe that it was optimizing intuition which made these students conclude and assert that it was not possible to get to number 25 in fewer steps.

c) The lack of argumentation in groups 5 and 10, even though they were instructed to justify their answers, suggests they used optimizing intuition, which made them conclude that their solution was the optimum.

d) Group 5 presented correct answers to both problems, a fact which makes us believe that they gave globally intuitive results because they used optimizing intuition.

From the analysis of answers we can conclude that our second prediction was fulfilled because most of the solutions showed an intention to apply the rule that demanded the justification of answers, and there was only one group whose answer could be considered as intuitive in both problems because there was no explicit argumentation, especially in problem 2. This group was No. 5, which was the only one with two members. Figure 14 shows this group’s solutions. In their solution to problem 1 this group did not offer any explicit explanation about the maximum nature of the obtained value, although we can discern an attempt towards justification (on the right of the figure) in the sense of a verification of all possible cases: they show all possible integer values for the lengths of the sides of the parallelogram. If we consider what is written on the right as an argumentation, then the intuition would not be in the answer but in a part of the argumentation. Specifically, the property “If a parallelogram has its vertices with integer coordinates and its perimeter is an integer number, then the lengths of its sides are integer numbers.” is considered as evident.



Maximum area = base * height

step

Figure 14. Group 5's solutions

Language: Terms and expressions: area, algebraic expressions, coordinates. Representations: Draw a parallelogram with two non-perpendicular sides and another one corresponding to the solution, placed in the coordinate system.
Situation-problem: Problem of geometric context.
Concepts: Area and perimeter.
Propositions: (Implicit) A square is a parallelogram. (Implicit) If a parallelogram has its vertices with integer coordinates and its perimeter is an integer number, then the lengths of its sides are integer numbers.
Procedures: They show possible integer values of the lengths of the sides of the parallelogram.
Arguments: None

Table 4. Cognitive configuration of the group solution to problem 1

Language: Terms and expressions: indicated operations, parenthesis.
Situation – problem: Arithmetic context problem.
Concepts: Subtraction and multiplication.
Propositions: None; *Procedures:* None; *Arguments:* None

Table 5. Cognitive configuration of the solution to problem 2

Although we conclude that there is no reason to rule out the presence of optimizing intuition in the answers of some groups it should be acknowledged that there are other possible explanations that do not require recourse to intuition. For example, the impact of many years of prior mathematical study should not be discounted. That is, even where contracts may be less explicit the impact of implicit beliefs about what is involved in mathematical solutions may be expected to exert some influence.

D'Amore, Font and Godino (2007) interpret the notions of didactic contract and didactic meta-contract within the framework of the OSA. The didactic meta-contract is understood as the set of rules that, during a long period of time, form part of any didactic contract. Some of these rules are clearly metarules, in the sense that they are rules that apply to other rules. An example would be the rule which says that rules must be abided by. On the other hand, there are other rules which form part of this meta-contract not so much because they can be considered as metarules (because they are rules about rules), but rather because they remain inalterable during a certain period of time (for example, during high school). This distinction raises a possible objection to the assumption that the students who took part in this research were immersed in a didactic contract that obliges them to justify their statements. Although the didactic contract appears to require such a justification it cannot be ruled out that the inertia of the meta-contract from previous years might lead them to consider that, in certain cases, justification was not necessary.

6. CONCLUSIONS

Our first prediction was that if optimizing intuition exists, then a considerable number of students would provide intuitive solutions to non-trivial problems. That is, in their cognitive configuration the language would be reduced to only that which is necessary to give the correct answer, while the properties, definitions and procedures would remain implicit even though the predominant characteristic would be that the block of argumentation would remain implicit or would be restricted to referring to the evidence. We consider that this prediction was fulfilled (see Figures 5 and 9). Given that there are different reasons which could explain the absence of justifications (for example, the students may not have known how to justify their solution), the next step should be to design new experimental situations which seek to falsify the hypothesis that there is an optimizing intuition.

Our second prediction was that if students working in groups solved the problems, there would be very few solutions without justification. This prediction was also fulfilled because in the majority of group solutions there are statements that try to justify the steps made. When solving problem 1, group 5 was the only one to show only its result, while when solving problem 2, only groups 5 and 10 showed only their results. Therefore, we can assert that the students of group 5 overcame the filter of the group solution and that their solutions could be considered as indicating the existence of optimizing intuition. Thus, metaphorically, we could say that they provided support to a "theorem of existence".

Our third prediction was that even in those cases in which the students' cognitive configurations present explicit arguments, some of the steps in the argumentation could also indicate the existence of optimizing intuition. This prediction was also fulfilled.

Our final conclusion with respect to the question "Is there an optimizing intuition?" is that there is some basis for giving a priori an affirmative answer to this question (see section 2.4), as well as the fact that the experimental situation designed to refute this supposition does not offer enough reasons to reject it. With respect to the design of the experiment that was conducted, one might object that even in the case in which we can be sure to have a didactic contract in which the results have to be justified, students will not make the effort to explain them if the tasks do not offer a "natural need" for justification. Taking the second problem as an example, we can acknowledge that because in a tree diagram all the cases can be shown, there is a case in which justification will add no insight, only a kind of verification that all the possible calculations have been taken into account but nothing else. When the student has all the calculations in front of him there is no real motivation to justify, because the solution is evident and the student doesn't see why it is necessary to argue that he has found the solution by using direct and exhaustive inspection in all the possible cases. Furthermore, it could be that students' years of socialization into what it means to do mathematics may be as powerful or more powerful than the explicit "didactic contract" in the immediate problem-solving situation.

Because these are important objections, the next step, already considered in the research that we are conducting to answer the research questions, is to design new experimental situations in which the tasks are selected in such a way that the justification is more clearly necessary than in the two tasks that were previously discussed, and where the need for a proof should be more than a mere consequence of the didactic contract. We will also conduct post hoc student interviews to interpret the meaning of written work.

With respect to the role of rigor and formalization in the solving of optimization problems by university students, our conclusion was that we perceive deficiencies in the use of propositions, procedures and arguments when solving the proposed optimization problems. There are cases in which these are not explicitly shown, as is illustrated in Figure 11 and in the cognitive configuration of the solution given by student 29 (Table 3). One specific deficiency in argumentation was the scarce presence of a justification that the result obtained was the optimum, a situation that can be more clearly seen in the solving of the discrete variation problem.

Finally, we want to highlight that even though there are many studies on intuition, rigor, the solving of problems and optimization, the relevance of this research is that we worked on all these notions simultaneously. The characterization of intuition as a vector made up of three components (idealization, generalization and argumentation) allows us to see the connections between intuition and rigor: on the one hand, intuition makes us regard some propositions as evident (the proposition could be the solution to the problem or an intermediate proposition in an argumentative sequence); and on the other hand, intuition plays an important role in the use of finitist reasoning and with generic elements.

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REFERENCES

Acevedo, J. I. (2008). Fenómenos relacionados con el uso de metáforas en el discurso del profesor. El caso de las gráficas de funciones. (Unpublished doctoral dissertation. Spain, Universitat de Barcelona).

D'Amore, B., Font, V. and Godino, J. D. (2007). La dimensión metadidáctica en los procesos de enseñanza y aprendizaje de la matemática. *Paradigma,* XXVIII(2), 49-77.

Descartes, R. (1986). *Meditations on First Philosophy: With selections from the Objections and Replies,* translated and edited by J. Cottingham. Cambridge, U.K.: Cambridge University Press.

Dubinsky, E. (2000). Meaning and formalism in mathematics. *International Journal of Computers for Mathematical Learning,* 5(3), 211-240.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht: Reidel.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics,* 24 (2), 139-162.

Fischbein, E. and Grossman, A. (1997). Schemata and intuitions in combinatorial reasoning. *Educational Studies in Mathematics,* 34, 27- 47.

Fischbein, E. and Schnarch, D. (1997). The evolution with age of probabilistic, intuitively-based misconceptions. *Journal for Research in Mathematics Education*, 28 (1), 96-105.

Fischbein, E., Tirosh, D. and Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics,* 10, 3- 40.

Font, V. and Contreras, A. (2008). The problem of the particular and its relation to the general in mathematics education. *Educational Studies in Mathematics,* 69, 33-52.

Font, V., Godino, J. D. and Contreras, A. (2008). From representations to onto-semiotic configurations in analysing the mathematics teaching and learning processes in L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in Mathematics Education: Epistemology, Historicity, Classroom, and Culture (pp. 157-173).* Sense Publishers: The Netherlands

Font, V., Godino, J. D. and D'Amore, B. (2007). An onto-semiotic approach to representations in mathematics education. *For the Learning of Mathematics,* 27(2), 1-7.

Font, V., Godino, J.D., Planas, N. and Acevedo, J. I. (2010). The object metaphor and synecdoche in mathematics classroom discourse. *For the Learning of Mathematics*, 30(1), 15-9.

Font, V., Planas, N., and Godino, J.D. (2010). Modelo para el análisis didáctico en educación matemática [A model for didactic analysis in mathematics education]. Infancia y Aprendizaje, 33(1), 89-105.

Fujita, T., Jones, K. and Yamamoto, S. (2004). The role of intuition in geometry education: learning from the teaching practice in the early 20th century. *10th International Congress on Mathematical Education (ICME-10*). Copenhagen, Denmark, 4-11 July 2004, pp 1-15.

Godino, J. D., Batanero, C. and Font, V. (2007). The Onto Semiotic Approach to Research in Mathematics Education. *ZDM-The International Journal on Mathematics Education*, 39(1-2), 127-135.

Godino, J. D., Batanero, C. and Roa, R. (2005). An onto-semiotic analysis of combinatorial problems and the solving processes by university students. *Educational Studies in Mathematics,* 60 (1), 3-36.

Godino, J.D., Contreras, A. and Font, V. (2006). Análisis de procesos de instrucción basado en el enfoque ontológico-semiótico de la cognición matemática. *Recherches en Didactique des Mathématiques*, 26 (1), 39-88.

Godino, J. D., Font, V., Wilhelmi, M. R. and Castro, C. de (2009). Aproximación a la dimensión normativa en didáctica de las matemáticas desde un enfoque ontosemiótico [An approach to the normative dimension in mathematics teaching from the onto-semiotic perspective]. *Enseñanza de las Ciencias*, 27 (1), 59-76.

Johnson, M. (1987). *The Body in the Mind.* Chicago: Chicago University Press.

Lakoff, G. and Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Linchevski, L. and Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics*, 39, 131-147.

Malaspina, U. (2007). Intuición, rigor y resolución de problemas de optimización. *Revista Latinoamericana de Investigación en Matemática Educativa*, 10(3), 365-399.

Malaspina, U. (2008). Intuición y rigor en la resolución de problemas de optimización. Un análisis desde el Enfoque Ontosemiótico de la Cognición e Instrucción Matemática. (Unpublished doctoral dissertation. Pontificia Universidad Católica del Perú).

Malaspina, U. and Font, V. (2009). Optimizing intuition. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, C. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 81 - 89. Thessaloniki, Greece: PME.

Núñez, R. (2000). Mathematical idea analysis: What embodied cognitive science can say about the human nature of mathematics, in T. Nakaora and M. Koyama M. (eds.). *Proceedings of PME24* (vol.1, pp. 3-22). Hiroshima: Hiroshima University.

Piaget, J. and Inhelder, B. (1963). *The child's conception of space.* London: Routledge and Kegan Paul.

Raftopoulos, A. (2002). The spatial intuition of number and the number line. *Mediterranean Journal for Research in Mathematics Education*, 1(2), 17-36.

Ramos, A. B. and Font, V. (2008). Criterios de idoneidad y valoración de cambios en el proceso de instrucción matemática. *Revista Latinoamericana de Investigación en Matemática Educativa*, 11 (2), 233-265.

Semandi, Z. (2008). Deep intuition as a level in the development of the concept image. *Educational Studies in Mathematics*, 68, 1-17.

Sirotic, N. and Zazkis, R. (2007). Irrational numbers: the gap between formal and intuitive knowledge. *Educational Studies in Mathematics*, 65, 49-76.

Tall, D. (2006). A theory of mathematical growth through embodiment, symbolism and proof. *Annales de Didactique et de Sciences Cognitives*, Irem de Strasbourg, 11, 195–215.

Tirosh, D. and Stavy, R. (1999). Intuitive rules: A way to explain and predict students' reasoning. *Educational studies in Mathematics*, 38, 51-66.

Tsamir, P. (2007). When intuition beats logic: prospective teachers' awareness of their *same sides – same angles* solutions. *Educational Studies in Mathematics*, 65, 255-279.

Tsamir, P. and Tirosh, D. (2006). PME 1 to 30 – summing up and looking ahead: a personal perspective on infinite sets' in J. Novotná, H. Moraová, M. Krátká and N. Stehlíková (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, Prague, pp. 49-63.

Wittgenstein, L. (1953). *Philosophical investigations*. New York: The MacMillan Company.