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The problem of the particular and its relation to the general in mathematics education

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Abstract Research in the didactics of mathematics has shown the importance of the problem of the particular and its relation to the general in teaching and learning mathematics as well as the complexity of factors related to them. In particular, one of the central open questions is the nature and diversity of objects that carry out the role of particular or general and the diversity of paths that lead from the particular to the general. The objective of this article is to show how the notion of semiotic function and mathematics ontology, elaborated by the onto-semiotic approach to mathematics knowledge, enables us to face such a problem.

Keywords Process of generalization and particularization · Generic element · Semiotic function

1 Introduction

Many of the difficulties observed in the teaching and learning of mathematics are related to the fact that in mathematical reasoning, to move from the general to the general it is necessary to pass through the particular. The mechanisms that language allows us for the particularization or individualization of mathematical objects are varied, as too are the processes of generalization (or abstraction). For example, Piaget (2001) distinguishes between reflective and empirical abstraction, while Radford (2003) distinguishes between

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three types of generalization, depending on the layer of generality and the kind of signs used to accomplish the generalization. These are factual, contextual and symbolic generalizations. Without doubt, the problem of the particular and its relation to the general makes up one of the central nuclei, not only of our discipline, but also of epistemology, psychology and other sciences and technologies that study human cognition, its nature, origin and development. The diversity of disciplines interested in this relationship is the reason for the diversity of approaches and ways of conceiving it.

In this article, we will look at the problem of the particular and its relation to the general in teaching and learning mathematics from the holistic viewpoint that underlies the onto-semiotic approach to cognition and mathematical instruction (Godino, Batanero and Roa 2005; Godino, Batanero and Font 2007). To be more precise, we are going to deal with the following problematic aspects of this relation:

1. The delimitation of the processes of particularization and generalization with respect to the processes of materialization and idealization
2. The elaboration of a typology of processes of generalization
3. The role that the generic element plays in the relation particular–general
4. The relation of the processes of generalization with other mathematical processes

In this first part of the article, the problem and the objectives are posed. Then, in the next section, we briefly present the theoretical framework of the onto-semiotic approach. In the third part, we study the delimitation of the processes of particularization and generalization with respect to the processes of materialization and idealization. In this section, a typology of processes of generalization is also proposed. In the fourth part, we reflect on the role of the "generic element" in mathematics and its relation with the duality particular-general. In the fifth part, we reflect on the relation of the processes of generalization with other processes. And, finally, in the last part, we present a synthesis of the response given by the onto-semiotic approach to the questions posed, ending with some general conclusions.

2 The onto-semiotic approach

The onto-semiotic approach to mathematical cognition tackles the problem of meaning and the representation of knowledge by elaborating an explicit mathematical ontology based on anthropological (Bloor 1983; Chevallard 1992), semiotic and sociocultural theoretical frameworks (Ernest 1998; Presmeg 1998a; Sfard 2000; Radford 2006a). It assumes a certain socio-epistemic relativity for mathematical knowledge since knowledge is considered to be indissolubly linked to the activity in which the subject is engaged and is dependent on the cultural institution and the social context of which it forms part (Radford 1997).

In Fig. 1 we represent some of the different theoretical notions of the onto-semiotic approach for mathematical knowledge (Godino and Batanero 1998; Godino 2002; Godino et al. 2005; 2007; Godino, Contreras and Font 2006a; Godino, Font and Wilhelmi 2006b). Here mathematical activity plays a central role and is modelled in term of systems of operative and discursive practices. From these practices the different types of mathematical objects (language, arguments, concepts, propositions, procedures and problems) emerge, which are related, building cognitive or epistemic configurations among them (hexagon in Fig. 1). Lastly, the objects that appear in mathematical practices and those emerging from these practices, depend on the language game in which they participate (Wittgenstein 1953), and might be considered from the five facets of dual dimensions (decagon in Fig. 1):

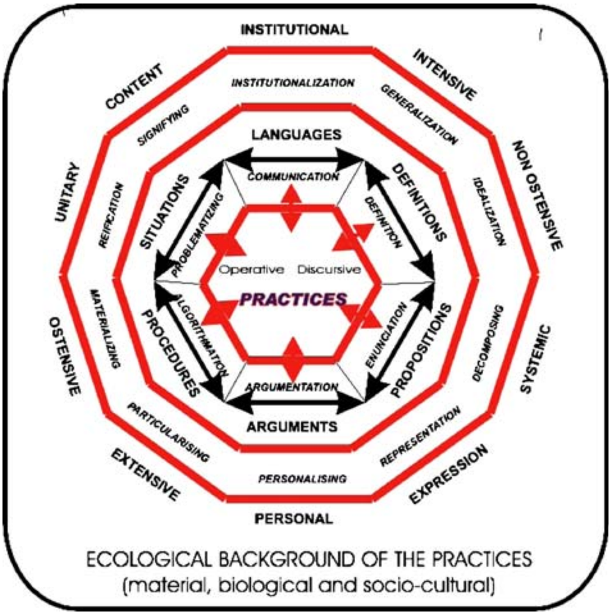


Fig. 1 An onto-semiotic representation of mathematical knowledge

personal/institutional, elemental/systemic, expression/content, ostensive/non-ostensive and extensive/intensive. The dualities, as well as objects can be analysed from a process-product perspective, which leads us to the processes in Fig. 1.

In the onto-semiotic approach the intention is not to give at the beginning a definition of "process", as there are many different types of processes: one can talk of process as a sequence of practices, as cognitive processes, metacognitive processes, processes of instruction, processes of change, social processes, etc. These are very different processes in which, perhaps the only characteristic many of them might have in common is the consideration of the *time* factor and, to a lesser degree, the sequence in which each member takes part in the determination of the following. For this reason, in the onto-semiotic approach instead of giving a general definition of the process, the selection of a list of processes considered important in mathematical activity is opted for (those of Fig. 1), without claiming to include in it all the processes implicit in mathematical activity, nor perhaps even the most important, for, among other reasons, some of the most important (for example, the process of understanding, the solving of problems or modelling) are more than processes; they are hyper- or mega-processes:

3 Materialization–idealization and particularization-generalization

The explanation of the difficulties students experience in understanding the objects and mathematical processes that relate the particular with the general, in our opinion, are to be found in the onto-semiotic complexity of the said objects and mathematical processes. Theoretical tools which allow for the clarification and differentiation of processes which in many cases, are presented under the heading of "generalization process" as if it were a single process, are necessary. It is not strange that the product of the process of generalization is presented, indistinctly, as an objectification, an idealization, a generalization, an abstraction, etc. To deal with this problem, in this section, first the processes of particularization and generalization are defined with respect to the processes of materialization and idealization. We then go on to propose a typology of generalization processes.

A good example to illustrate the complexity associated with the move from the particular to the general is to be found in the book *Geometry with Applications and Proofs* (Goddijn, Kindt and Reuter 2004), edited by the Freudenthal Institute. In the first task set to the students they are presented with a map of a desert with five wells and they are asked to imagine they are a shepherd who has a flock of sheep in a specific position in this desert. They are then asked questions like "which well would you go to for water?" (Goddijn et al. 2004, part I) The task given to the students is an extra-mathematical situation the resolution of which allows the emergence of, among other things, a new mathematical object: the partition of an area according to the *nearest-neighbour-principle.* The authors intend to present a situation of an extra-mathematical context that is understood by the student as a particular case of a mathematical object. In this case, the particular is extra-mathematical and the general is a mathematical object.

The detailed analysis of the activity necessary to solve the task shows that many of the processes in Fig. 1 are put into play, but we do not go into detail here because of questions of space. We will limit ourselves to pointing out that the task we are analysing is divided explicitly into two parts: (1) The statement of the problem and (2) the commentary that follows to the problem, the text being as follows:

> In this exercise you just partitioned an area according to the nearest-neighbour-principle. Nowadays similar partitions are used in various sciences, for instance in geology, forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few. We will revisit those now and then. Next we will investigate the simple case of two wells, or rather, two points, since we might not be dealing with wells in other applications. (Goddijn et al. 2004, part I, p. 5)

If we look at this from the perspective of the processes considered in the onto-semiotic approach, the statement of the problem aims to generate a process of personalization (in the sense that the student constructs, among other things, the mathematical object partition of an area according to the nearest-neighbour-principle). On the other hand the later commentary tries to institutionalize this mathematical object, in the sense that it is something known by all the class, that is to say, it comes to exist as a mathematical object in the classroom

To try to achieve this institutionalization, in the quotation above, first the authors generate a hidden process of idealization (the desert becomes an area) and then they establish the particular-general relationship between the idealized situation and the mathematical object, "In this exercise you just partitioned an area according to the nearest-neighbour-principle". The way of presenting the general to the student is the result of an additive abstraction, as it consists of a coming together of different elements in the same group, "Nowadays, similar partitions are used in several sciences, for instance in geology, forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few." We then go on to a process of particularization by saying, "Next we will investigate the simple case of two wells", and one of explicit idealization, when the wells are converted into points, "or actually two points, since we might not be dealing with wells in other applications".

3.1 Processes of idealization and materialization

The processes of materialization–idealization in the onto-semiotic approach are associated with a dual dimension or ostensive/non-ostensive facet. Mathematical objects are, in general, non-perceptible. However, they are used in public practices through their associated ostensives (notations, symbols, graphs, etc.). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

Let us suppose that the teacher has drawn on the board the figure on the left (Fig. 2) and talks about it as if it was the perpendicular bisector of the segment that has at its extremes the points *A*(3,4) and *B*(6,2) expecting the students to interpret the figure in this way.

If we look carefully at Fig. 2 on the left one observes that: (1) the segment is not a segment of a straight line, (2) the bisector is not a straight line as it is only a segment of the bisector, (3) neither is the said segment a segment of the straight line, (4) it does not pass exactly through the midpoint, (5) the points *A* and *B* and the midpoint are very thick, the angle that forms the supposed midpoint with the segment is not exactly 90°, etc.

It is clear that the teacher hopes the students will go through the same process of idealization of the figure on the board as he has done and that his discourse leaves out the problems commented on in the previous paragraph. That is to say, the figure on the blackboard is an ideal figure, explicitly or implicitly, for the type of discourse the teacher makes about it. Figure 2 on the board is concrete and ostensive (in the sense that it is drawn with chalk and is observable by anyone who is in the classroom) and as a result of the process of idealization one has a non ostensive object (the perpendicular bisector of the segment *AB*) in the sense that one supposes it is a mathematical object that can not be presented directly if it is not by means of certain associated ostensives. On the other hand this non ostensive object is particular, to know, it is the perpendicular bisector of the segment of extremes *A*(3, 4) and *B*(2, 6) and it is not, for example, the perpendicular bisector of the extremes (4, 4) and (8, 7). This type of "individualized" object in the ontosemiotic approach we call an extensive. Therefore, as a result of the process of idealization we have passed from an ostensive which was extensive, to a non ostensive that continues to be an extensive.

The other side of the coin is that to be able to manipulate non ostensive objects we need ostensive representations which are the result of a process of materialization (and also of

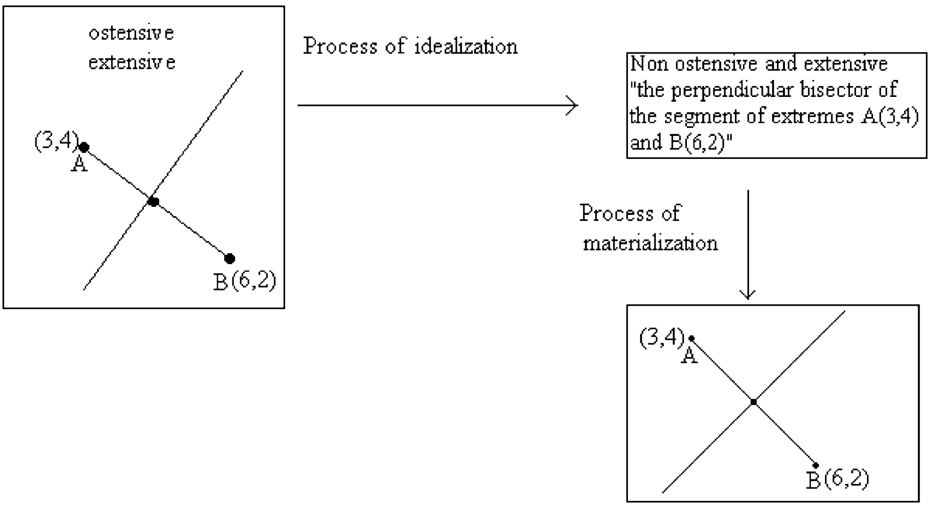


Fig. 2 Processes of idealization and materialization

representation). Continuing with the example of the perpendicular bisector drawn on the board, the teacher might realise that Fig. 2 is not well drawn and then erase it and substitute it for a more perfect figure (Fig. 2 on the right). The process of materialization places the mathematical knowledge in the "territory of the artefact" (Radford 2006b, p.107), as its products are cultural artifacts which influence and materialize thought.

The process of idealization is one which duplicates entities because, as well as the ostensive which is in the world of human material experiences, it is believed (at least in a virtual way) an idealized non ostensive is created. The relation that is established between these two entities is the content–expression and one can fall into the error of segregating this pair of objects and give an independent life to the non ostensive objects (something similar to when the spirit is considered as something separate from the body), among other reasons because the discourse about the object that is usually used in mathematics leads one to believe in the "existence" of the mathematical object as something independent from its representation. It was probably Wittgenstein (1978) who most notably drew attention to this danger: for this philosopher the assimilation of names to mathematical terms, especially the conception that they are names of ideal or abstract objects, is fundamental in the confusions that are produced on thinking about mathematics.

3.2 Processes of particularization and generalization

The processes of particularization-generalization in the onto-semiotic approach are associated with the extensive/ intensive facet. An extensive object is used as a particular case (a specific example, i.e., the function $y=2x+1$), of a more general class (i.e., the family of functions $y=mx+n$), which is an intensive object. The terms extensive and intensive are suggested by the two ways of defining a set, by extension (an extensive is one of the members of the set) and by intension (all the elements are considered at the same time). By extensive we understand a particularized object (individualized) and by intensive, a class or set of objects.

The mechanisms that language offers us for the particularization or individuation of mathematical objects are varied (for example, the grammatical deictics: this, that, here, there, etc., or the indefinite determinants: a/an, some, any, etc.) The processes of generalization (or abstraction) which permit intensives are also varied. In the onto-semiotic approach, three types of process are considered: reflective or constructive abstraction, the eliminative and the additive.

Reflective abstraction is a process, which apart from the reflection on the system of actions and their symbolization, finds invariant relations and describes them symbolically. This means that in this process, certain properties and relations are noticeable and attention is focused on them, which shows that they attain a certain degree of independence with respect to the objects and situations they were initially associated with. This type of abstraction produces a result that comes out as action and acquires sense and existence from it. One of the characteristics of reflective abstraction is that it is constructive, in the sense that it constructs intensives from the reflection on the action.

Now we can consider other different mechanisms for obtaining intensives, one of an eliminative type and the other an additive. Empirical abstraction functions by means of an eliminative mechanism; this involves eliminating or separating aspects or specific notes. In this case one arrives at an intensive by the basic application of the type/example relation, which is based in the application of a mechanism of an eliminative type on the basis of the part/whole relation, that is to say the intensive (type) is considered as one of the parts which make up the extensive (whole) as the latter is a specific example that contains many notes or different attributes.

Another different mechanism for obtaining intensives consists in the coming together of different elements in one set. For example, we can consider the perpendicular bisector of Fig. 2 as a member (an extensive) which forms part, together with other perpendicular bisectors, of a class or set (an intensive). In the latter case one also arrives at an intensive because of the whole/part relation, but it is understood in an inverse way to how one understands the case of empirical abstraction, the part (the extensive) is a member of a whole, a class (the intensive).

These three ways of generating intensives play a different role in mathematics. The eliminative and the constructive would have to be seen, above all, within the "context of discovery", while additive abstraction is related, above all, with the "context of justification", given that the latter is usually used in the formalist presentation of mathematics which is based on the theory of sets.

4 The generic element

Mathematical reasoning, going from the general to the general, introduces an intermediate phase that consists of contemplating an individual object. This fact poses a serious dilemma: if reasoning has to be applied to a specific object (for example a triangle), it is necessary for there to be some guarantee that we reason about any object so that it is possible to justify the generalisation in which the reasoning ends. Furthermore, since the specific object is associated with its representation the problem of whether the representation refers to a specific object or to a general concept, appears (Font, Godino and D'Amore 2007).

The introduction of the extensive/intensive and the expression/content dualities in the onto-semiotic approach can help to clarify the problem of the use of generic elements (Contreras, Font, Luque and Ordóñez 2005). These two dualities become an essential instrument in analysing the complexity associated with the use of the generic element. Expressed differently, the use of the generic element is associated with a complex network of semiotic functions (and so representations) that relate intensive with extensive objects. We will demonstrate this with the example of the definition of derivative function.

4.1 The definition of derivative function as a context of reflection

As a context of reflection we are going to use the following definition of derivative function which is found in a book for Spanish Bachillerato students (16–17 years old):

> Task 1.
>
> 1. Derivative function.
>
> We consider now, given a function $y=f(x)$, another new function that associates at each point of the domain of f its derivative f'(a) when this exists. This function is the derivative function of $y=f(x)$ and is represented with $f'(x)$ or y'.
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In the text book, before this definition, the derivative of the function $y=f(x)$ at $x=a$ has been defined.

4.2 Thesis

We will go on to develop the following thesis:

Thesis In order to understand the previous definition, a hypothetical student has to put into practice (plausibly) a series of semiotic functions like those we describe in Fig. 4.

In this thesis to start with we interpret the understanding of an object *O* by a subject *X* in terms of the semiotic functions that *X* can establish, in some fixed circumstances, in which *O* intervenes as expression or content.

The first thing we will do is to consider the extensive/intensive facet of mathematical objects as expression or content. More specifically, we use the following typology of semiotic functions (from row to column; Table 1):

1. SF1 This semiotic function relates an extensional entity with another extensional entity.
   - SF1.1 relates an object with another of the same class.
   - SF1.2 relates an object with another which is not of the same class.
2. SF2 This semiotic function relates an extensional entity with an intensional entity.
   - SF2.1 relates an object with the class it belongs to.
   - SF2.2 relates an object with a class it does not belong to.
3. SF3 This semiotic function relates an intensional entity with an extensional entity.
   - SF3.1 relates a class with an example of the class.
   - SF3.2 relates a class with an object which is not of that class.
4. SF4 This semiotic function relates an intensional entity with another intensional entity.
   - SF4.1 This semiotic function defines a class of objects in a different way.
   - SF4.2 This semiotic function relates an intensional entity with another different intensional entity.

What these semiotic functions have in common is that all of them are of the representational type, in the sense that they facilitate the representation of the expression of the content, but also they can be of different types according to whether the expression or the content are extensive or intensive and according to what the correspondence criterion is between the expression and the content. For example, on one hand SF2.1 is representational, in the sense in that the particular case can be taken as a representative of the class, but on the other hand it is of the metonymic type – in particular a synechdoche (Presmeg 1998b) – as an extensive (a part) is taken for the whole (the class), in this case the correspondence criterion is that of "belonging". In some cases the semiotic functions are exclusively representational as we shall see later on.

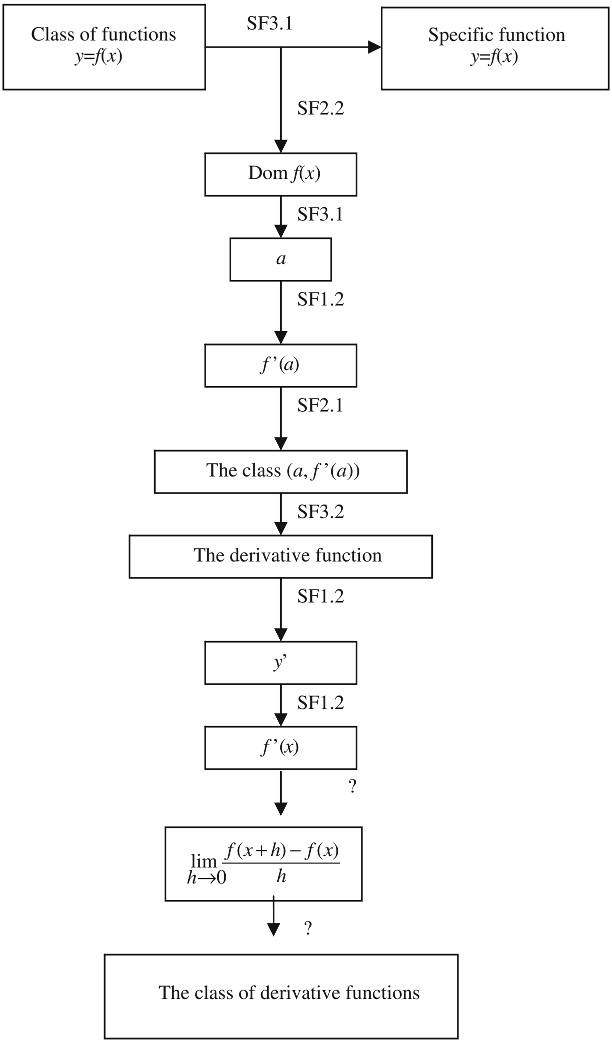
Table 1 Types of semiotic functions

	Extensional	Intensional
Extensional	SF1	SF2
Intensional	SF3	SF4

The application of this type of semiotic function to the understanding of the definition of the derivative function allows us to affirm, in the first place, that the author of the text assumes that the hypothetical student puts into practice a series of semiotic functions like those described in Fig. 3.

SF3.1: This semiotic function indicates that, from the class of all the functions, a specific function is considered $y=f(x)$.
SF2.2: Relates an object (the function) with a class to which it does not belong (its domain).
SF3.1: Relates the class (domain) with an element of the said class (the value a).
SF1.2: Relates an extensive (a) with another ($f'(a)$).
SF2.1: Relates the pair (a, f' (a)) with the class of the pairs (a, $f'(a)$).
SF3.2: Relates the class of pairs (a, f' (a)) with the derivative function object.

Fig. 3 Series of semiotic functions



SF1.2: Is a semiotic function of the representational type that relates the extensive derivative function with the extensive y'.

SF1.2: is a semiotic function of the representational type that relates the extensive y' with another extensive $f'(x)$

The last two semiotic functions of Fig. 3 (with a question mark) are not explicitly given in the text and, therefore, they are left up to the student. This may give rise to semiotic conflicts. The last one, which is of the type SF2.1, is necessary so that the student understands that the derivative function obtained from the function $y=f(x)$ is a member of the class of derivative functions.

It appears even more serious to us to leave the penultimate semiotic function, which permits the interpretation of $f'(x)$ as $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to the student. To our understanding, this semiotic function can only be carried out if the student has completed the previous stage with the semiotic functions that appear, later in Fig. 4 (marked with asterisks).

SF1.2*: Of representational type and relates an extensive with another extensive.

SF2.1**: Relates an element with the class to which it belongs.

If the author of the analysed text has not designed any sequence of activities to prevent this potential semiotic conflict, it should be pointed out that there are other authors who are

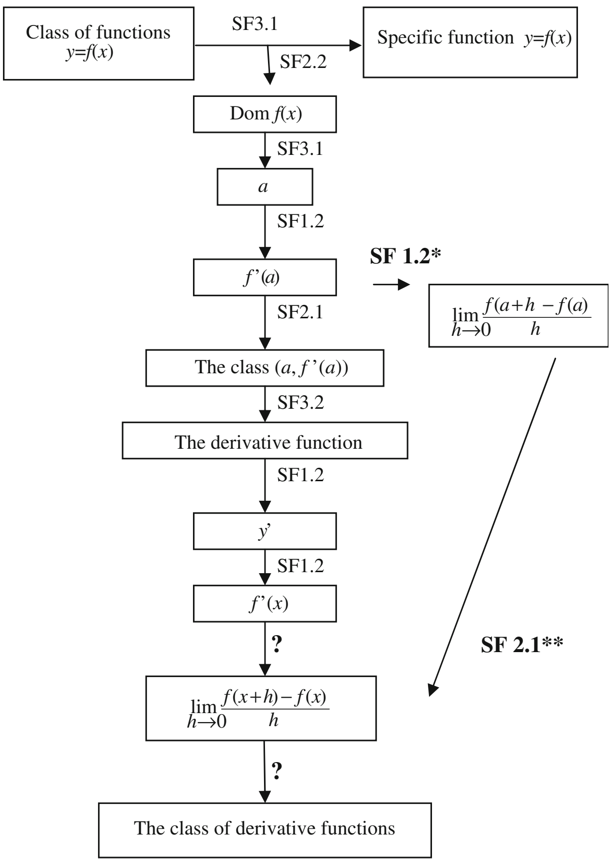


Fig. 4 Expanded series of semiotic functions

conscious of this. There are manuals, corresponding to the same course, which after having defined the derivative function at a point as a limit and before going on to define the derivative function as a limit, design a didactic sequence that aims to facilitate the understanding of the pair $(a, f'(a))$ as an element of a class – the case of the function $f(x)=x^2$, for example – which can be understood as a function, called the derivative function, that allows a value to be found for each derivative function. This new function is useful in saving the calculation of $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ for each value a given that it is possible to obtain it by another method.

This sequence helps the student to understand that the image $f'(x)$ allows us to obtain the limit $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, and thus facilitates the association of the expression $f'(x)$ with the content $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

4.3 Justification of the thesis

Having shown in Fig. 4 the complete set of functions that we consider the student must activate in order to understand the definition of the derivative function we are analysing, we move on to justify the plausibility of the procedure.

If the semiotic analysis we have just made is "a priori" and serves to detect hypothetical semiotic conflicts, we want to point out that the eight semiotic functions we have used in the detailed description cannot be separated from the reflection we have made on the crucial elements of the mathematical activity: the use of generic elements and the observation of episodes in the classroom in which the rules of use are determined. It should be pointed out then, that to prove that understanding can be described by means of a set of semiotic functions, we also have to understand comprehension as competent use deriving from knowledge and the application of the rules.

So then, in our opinion, with relation to the generic element, it is necessary to consider three different but interconnected questions, in order to know:

1. Why, in the demonstration of a mathematical proposition (the enunciation of a definition, etc.) does one introduce an intermediate phase which refers to a particular object?
2. How is it possible that a reasoning in which this phase intervenes, can, in spite of this, give rise to a universal conclusion?
3. The particular element normally forms part of a chain in which the previous links are generic elements. At the same time, the particular element on being considered generic will be converted into the previous link of a new case and so on successively.

With respect to the first question, different solutions can be given, as for example that which Descartes proposes in his fifth meditation (Descartes 1986): it is necessary to consider a specific object for the intuition, which cannot refer to itself but to particular objects, to be able to act. In this work, we do not intend to answer this question, we just affirm that when, in the definition of the derivative function of Section 4.2, one says "given a function $y=f(x)$", one has to take into account that the author aims to give the definition of the derivative function of any function. For this reason, the first thing to do is to direct the attention of the student to "a function", that is to say, to go from the general to the specific, and for this reason, a particular object is introduced by means of a intensive/ extensive semiotic function (an SF3.1). If the function is a particular case (an extensive) so too is its domain (a particular set).

We also consider that for a student to be able to understand the definition of derivative function in a textbook, he has to have the capacity to understand that the domain, on one hand is "a" set (an extensive) and on the other hand is a group formed of different values (an intensive). We also assume that the student understands the relation between the whole and the parts to progress in the definition of the derivative function of the textbook. We have synthesised this process in SF2.2, even though one could be more specific and suppose that previously there is a SF1.2 which relates a particular function (an extensive) with another particular object: its domain (another particular but different object). We have limited ourselves to putting the SF2.2 in the scheme because it seems to us the most relevant.

When in the definition it says "…that associates at each point to that of the domain of f…" we also consider that the previously mentioned phenomenon is produced again: the need to introduce an intermediate phase into mathematical reasoning that consists of the contemplation of a particular object, for this reason we have considered it a semiotic function of the type SF3.1.

In the definition it says: "…. that associates with each point of the domain its derivative $f'(a)$ when it exists "for a specific value to which corresponds another specific value, $f'(a)$". For this reason, we have considered it a semiotic function of the SF1.2 type.

Up till now the semiotic functions mentioned deal fundamentally with the introduction of the specific element in mathematical reasoning. From now on the second aspect appears which we have commented on before: How is it possible that a reasoning in which a similar intermediate phase intervenes, can, nevertheless, give rise to a universal conclusion?

With reference to this second question, the onto-semiotic approach has a clear position, even if too general: we act on a particular object, but we put ourselves in a "language game", in which it is considered that when we refer to this particular object, it is understood that we are interested in its general characteristics and that we leave out particular aspects. This affirmation is too general, as it does not give details of the characteristics of this language game and the difficulties students have participating in it.

The analysis of dialogues between teachers and students taken from different processes of the study has allowed us to detect some of the characteristics of the said language game. Here follow two dialogues in which each teacher explains to his students the rules that govern the use of the generic example.

4.3.1 Dialogue 1

In this dialogue the teacher asks the students to solve the following activity in their course book:

> Exercise: Given the function $f(x)=ax+b$, show that $f'(x_0)=a$, independently of the value x_0 considered.

This activity is set just after the teacher has explained in class a paragraph in the textbook in which the derivative of the constant function $f(x)=k$ is $f'(x)=0$ is justified, first graphically, reasoning on the slope of the tangent line at any point of the straight line, and after calculating the limit of the average rate of change of the function f.

> Teacher: You are going to do it in two different ways: graphically and using limits, OK? Come on then, and then someone will come out to the blackboard and correct it. Meanwhile I will be giving out some materials which will be useful afterwards, and so you will have it ready.

> Student (Iván): But graphically, we can…. This is an example, if we represent it graphically it is an example…

Teacher: (while nodding and confirming the student is right and approaching him) Yes, that's it!

Iván: And it says that x zero is considered….

Teacher: Yes, but Iván, to be able to prove it, take any point on this straight line, any one at all, and do it, and as you can do this with any point and with any straight line, it will help to prove it. OK? But you are right, of course, to be able to draw you have to choose a specific point and a specific line (while the teacher is talking she is giving out paper to the students).

4.3.2 Dialogue 2

After the teacher has introduced in previous classes the derivative function at a point as $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ and just after having introduced the derivative function as $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ the following dialogue was produced:

Student (Laura): What difference is there between the definition of the derivative function and the definition of the derivative function at a point?

Teacher: the derivative function at a point is $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, in this expression the *a* is fixed, it does not change, what does change is the *h*. By contrast, in the case of the derivative function $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, first one has to suppose that the *x* does not vary and that only the *h* varies in order to obtain *f'*(*x)*, and then one has to suppose that the *x* varies. So when you calculate the derivative function at a point the result is a number, while when you calculate the derivative function, the result is a formula of a function.

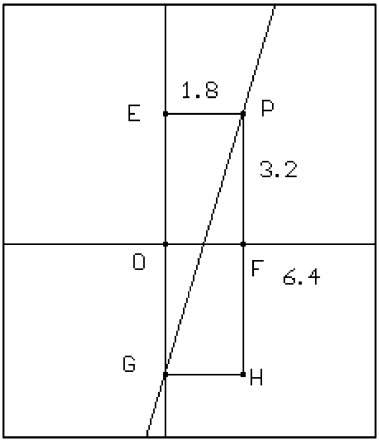
These dialogues show that understanding has a social dimension, illustrated in the dialogues by the role of other individuals, in this case the teacher. This vital aspect of understanding is contemplated in the onto-semiotic approach given that it is considered that the mathematics class constitutes a micro society where the construction and diffusion of mathematical knowledge takes place through the social interaction between the students and the teacher. As a consequence, mathematical learning is conditioned by different mathematical and didactic meta-knowledges, as are some of the socio-mathematical norms and clauses of the didactic contract. In these dialogues as a minimum we can observe (1) epistemic norms, (2) norms that regulate interactions, (3) meta-epistemic norms that regulate the use of generic elements and (4) norms that regulate the use of materials in the classroom (dialogue 1). Knowledge of meta-epistemic norms that regulate the use of generic elements is essential for students to be able to activate the series of semiotic functions in Fig. 4. A full treatment of these elements would require more space than is available here.

Finally we consider the production of the student (answer to the task 3) which shows that the student is conscious of the rules of use of the generic element as in her answer she takes into account the calculation of the derivative function.

4.3.3 Student production

The following task was given to the students to carry out using the "Cabri" software: Using the proposed construction (see Fig. 5) the students had first to conclude that the line which

Fig. 5 Task 2



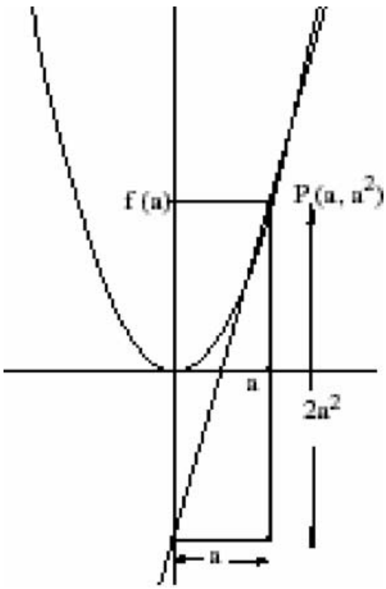
resulted from moving the point *P* was the parabola $f(x)=x^2$ and that the line *PG* was the tangent line to this function at point *P*. They also had to discover an invariant of the type: in the parabola $f(x)=x^2$ the tangent line at *P* cuts the axis of ordinates at a point such that the length of the segment that has for extremes this point and the origin of the coordinates is the ordinate of *P*.

They were then asked to use this property to carry out the following task:

Task 3

(a) If $OF=a$, prove that $GH=a$, $PF=a^2$ and $PH=2a^2$.
(b) If the derivative of the function at a point is the slope of the tangent line, calculate $f'(a)$.
(c) Prove that the derivative of the function $f(x)=x^2$ is $f'(x)=2x$ (Fig. 6).

Fig. 6 Task 3



We observe that, with the small letter "p" the student indicates the slope of the tangent line.

We see that, in the answer to section (c), the equivalence "$a=x$" is expressing that the reasoning of the sections (a) and (b) are valid for any value of a. This indicates that the student has entered into the language game that governs the use of generic elements (Fig. 7).

If we consider again the definition of the derivative function that we are analysing, it is worth pointing out that the phrase "another new function that associates with each point in the domain of f its derivative $f'(a)$ when this exists" puts us directly within the second aspect that we have commented on earlier: what has been said for the particular case is valid for any other particular case. The student has to understand firstly, that what one does with a specific value of the domain, can be done with all the values of the domain (an SF2.1, as we move from the particular case to the general) and then he or she has to understand that the class of pairs $((a, f'(a))$ can be considered as a particular object, a new function (an SF3.2).

When in the definition it says: "this function is the derivative function of $y=f(x)$ and is represented by $f'(x)$ or y'" we have different ways of representing the same object. The representations are on one hand, different, which justifies the use of the semiotic function SF1.2 that relates an extensive with another different extensive. But, on the other hand, they are representations of the same object. In this case we have semiotic functions which are exclusively representational.

When in the definition, the following expression is introduced $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ we enter into the third aspect we have commented on with relation to the generic element, as both x as well as h are generic elements. The semiotic functions that we propose for the correct understanding of this expression on the part of the student have been suggested by the analysis of the authentic dialogues recorded on video in different classes (see Dialogues 1 and 2). In these dialogues the teachers confront the problem of the simultaneous presence of various generics introducing different phases: but first it is necessary to consider as generic only one of those which intervene and then it is necessary to move on and consider as generic that which had been considered as non generic.

In the analysis we carry out with semiotic functions, on one hand we take into account the previous dialogues, and on the other, we assume that the student has already studied the

Fig. 7 Student's answer

The student's answer is:

a $GH = a$ because it is the same distance

$PF = a^2$ *because* the image of a in the function $f(x) = x^2$ is a^2

$PH = 2a^2$ because it is double FP

b $p = \frac{2a^{2}}{a} = 2a$

$f'(a) = 2a$

c $a = x$

$p = \frac{2x^{2}}{x} = 2x$

$f'(x) = 2x$

derivative function at a point. For this reason first we propose the semiotic function SF1.2* of the representational type that relates the object $f'(a)$ with the object that represents $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$. We assume that it is known that h is generic. That is to say the student establishes without difficulty a semiotic function that relates the number $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ with the type of values of the average rate of change of the function $f\frac{f(a+h)-f(a)}{h}$ when $h\to 0$ (this step does not appear in the stages of semiotic functions because, as we have already said, we assume that the student already understands the definition of the derivative function at a point when confronted by the analysed text). Next, he has to consider the value a as generic, this means he or she has to carry out the semiotic function SF2.1** that relates an element with the class to which it belongs. Finally, the student has to understand that the derivative function obtained through the function $y=f(x)$ is a member of the class of derivative functions (he or she has to carry out a SF2.1).

4.4 Some conclusions

The thesis we have justified in this fourth part allows for the explanation of a relevant didactic phenomenon related to the derivative object detected in different investigations: the regularity with which semiotic conflicts manifest themselves in practical lessons with students when they have to distinguish the derivative at a point of the derivative function. Having completed the analysis, we are in a position to affirm that the reason is that little importance is given to the semiotic complexity that the stage implies in the derivative at a point, to the derivative function in the didactic units intended for the level of the Spanish Bachillerato (16–17 years old).

The analysis of other texts, similar to that carried out previously for task 2, permits the conclusion that certain uses of the notation $\Delta y/\Delta x$ can present more problems than advantages when the semiotic complexity associated with the step from the derivative at a point to the derivative function is taken into consideration. Certain ways of introducing incremental and differential notation lay the foundations of a semiotic conflict caused by the implicit introduction of the derivative function in the definition of the derivative at a point. That is to say certain uses of incremental notation imply the definition at a point $f'(a)$ as: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ at $x=a$ without previously having defined the derivative function (Contreras et al. 2005).

The ontological semiotic analyses like those we have carried out here on the derivative function (and especially, the use of the duality intensive/extensive for the analysis of the use of the generic elements) are microscopic analyses which we consider to be complementary to other macroscopic analyses because they allow us to see aspects that the latter do not allow us to see. Put in another way, if a study of the techniques of derivation is carried out, its field of application, its modifications, and so on, but one does not do this finer type of semiotic ontological analysis, one runs the risk of "seeing" the calculation of the derivative function only as a slight modification of the technique used to calculate the derivative at a point, one is not conscious of the semiotic complexity that the stage of the derivative at a point implies with regard to the derivative function.

5 Relation of the processes of generalization to other mathematical processes

In writing up the questions for task 3 much attention has been paid to the step from the particular to the general. Without entering into a detailed analysis as carried out previously for task 1, in order to calculate the derivative function of a condition that satisfies all the tangents, the student has to identify the following network of semiotic functions:

1. Treat the variables related by the formula and the graph of $f(x)=x^2$, separately. To do this, it is necessary to understand this function as a process in which other objects, one being x and the other being $f(x)$, intervene. Here, a semiotic function that relates the object $f(x)$ to the object x, is established, having an instrumental role.
2. Associate x to the slope of the tangent line for the point on the x axis. This relation can be considered as a semiotic function that relates the object x with the object slope of the tangent line to the point on the x axis.
3. Associate the expression that permits us to calculate the slope of the tangent line to the point of the x axis with $f'(x)$. In this case, we have a semiotic function that relates one notation with another different but equivalent one.
4. Consider x as a variable. In this case, we have a semiotic function that relates an object to the class it belongs to.
5. Understand the function obtained as a particular case of the 'derivative function' class. In this case, we have a semiotic function that relates an object to the class it belongs to.

If we look at the task 3 handed to the students we can observe that the sequence of sections aims at making the establishment of these semiotic functions easier. The use of the letter a, in question b of the worksheet, has the role of introducing a specific element in the student's reasoning and so makes step 1 easier. The reason for including the use of the graph and the symbolic notation together for the point of coordinates (a, a^2) is that the teacher wants the students to carry out steps 2 and 3. Steps 4 and 5 are intended to be achieved from question c.

This example permits us to shed light on a phenomenon that we consider to be very relevant: the student, in order to carry out the majority of mathematical practices, has to activate a network of complex semiotic functions and the ostensive objects used are the means both to reduce or increase the complexity of this network or to carry out the practice correctly. For example, if we had eliminated question b and Fig. 6 in the worksheet, we would still want the student to apply the technique to calculate the derivative function and we would still use graphs (the ones from the previous activity with the computer) and symbolic expressions (question c). However, the complexity of the semiotic functions that the student would have to carry out would increase considerably and so also would the possibilities of not solving the task.

On the other hand, this student previously in task 2, using a dynamic software programme, found a condition that is satisfied by all the tangents. The emergence of this property is the result of a reflective abstraction. It should be pointed out that one arrives at an intensive generalization (which does not vary) from (1) overlooking aspects of the specific (which does vary) and (2) considering that what is valid for an object (variable in this case as we are working with a dynamic programme) is valid for all, which means one reasons with generic elements (in this case dynamic). In addition processes of idealization and materialization are present as it is considered that the graphic on the computer screen is a materialization of the ideal mathematical object "$f(x)=x^2$".

Actions that allow us to find the invariant that is common to all the tangents is carried out with dynamic software. The consequence is that the process of reflective abstraction is related to metaphoric processes as the graph of the function is structured implicitly in terms of the following metaphor: "The graph of a function can be considered as the trace of a point that moves over a path (the graph)" (Bolite, Acevedo and Font 2006) (Table 2).

Table 2 The graph of a function is the trace of a point that moves over a path (the graph)

Source domain	Target domain
Path	Graph of a function
Location on the trajectory at a given time	Point of the graph
Source Location	Source of the graph (for example, $-\infty$)
End of the path	End of the graph (for example, $+\infty$)
Out of the path	Points that don't belong to the graph
. . .	. . .

The fact that, for the different points that are obtained when moving the mouse (in Fig. 5), the name of the point is always *P* has a very outstanding power in this metaphoric projection.

6 Synthesis and conclusions

In this paper we have described some aspects of the problem of the particular and its relation to the general in teaching and learning mathematics and we have given a reply from the theoretical framework that we name the onto-semiotic approach.

With respect to the problem of the delimitation of the processes of particularization and generalization related to the processes of materialization and idealization, our conclusion is that the consideration of the dual facets in the onto-semiotic approach, especially the ostensive/non-ostensive and extensive/intensive facets allow one to deal separately with the processes of materialization and idealization and particularization and generalization. This is an important distinction as it permits a more detailed analysis, and consequently a better comprehension of each of these processes as well as of their combined presence in mathematical activity.

When we use an ostensive as a generic element in mathematical practices we are acting on a particular object, but we situate ourselves in a “language game” in which it is considered that when we refer to this particular object, it is understood that we are interested in its general characteristics and we disregard the particular aspects. In the case used as the context of reflection, the definition of the derivative, control of the rules of the game allows the student to activate a complex network of semiotic functions which is what produces the understanding of the said definition. The knowledge of the network is also useful in explaining the students’ difficulties. Therefore, we show how an onto-semiotic approach can help us analyse mathematical texts and thus help us avoid stumbling blocks and understand students' conceptual problems.

We refer to the fact that in the onto-semiotic approach, we propose that the expression and the content of a semiotic function can be any type of object, filtered by the remaining dualities, which provides a greater analytic and explanatory capacity. Furthermore, the type of relations between expression and content can be varied, not only be representational, e.g., “is associated with”; “is part of”; “is the cause of/reason for”. This way of understanding the semiotic function enables us great flexibility, not to restrict ourselves to understanding ‘representation’ as being only an object (generally linguistic) that is in place of another, which is usually the way in which representation seems to us mainly to be understood in mathematics education.

Finally, in this article we have set out the processes of generalization, which apart from being related to representational, idealization and materialization processes, can also be closely related to other processes such as, for example, metaphorical processes which we have not analyzed in detail in this paper.

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