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Metaphors in mathematics classrooms: analyzing the dynamic process of teaching and learning of graph functions

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Abstract This article presents an analysis of a phenomenon that was observed within the dynamic processes of teaching and learning to read and elaborate Cartesian graphs for functions at high-school level. Two questions were considered during this investigation: What types of metaphors does the teacher use to explain the graphic representation of functions at high-school level? Is the teacher aware of the use he/she has made of metaphors in his/her speech, and to what extent does he/she monitor them? The theoretical framework was based on embodied cognition theory. Our findings include teachers' expressions that suggest, among other ideas: (1) orientation metaphors, such as "the abscissa axis is horizontal"; (2) fictive motion, such as "the graph of a function can be considered as the trace of a point that moves over the graph"; (3) ontological metaphors; and (4) interaction of metaphors. We also show that teachers were not aware of using metaphors.

Keywords Cartesian graphs · Metaphor · Image schema · Discourse

1 Introduction

Understanding and using Cartesian graphs is a crucial component of high school mathematics and science courses. The research reported in this paper focused on the following two questions: What types of metaphors do teachers use to explain the graphic representation of functions at high school level[1]? Are teachers aware of the use they have made of metaphors in their speech, and to what extent do they monitor them?

[1] *Bachillerato* in Spain (17-18 years old)

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The paper is divided into five subsequent sections. The first section contains an introduction and comments on the research problem. The second briefly reviews the research on metaphor and presents the theoretical frameworks of embodied cognition theory. The third presents the study and its methodology. The fourth contains the data analysis regarding the two questions that constitute the goal of the research. Finally, in Section 5, we offer some conclusions.

2 Image schemas and metaphorical projections

In recent years, several authors (see, for instance, Acevedo, 2008; Bazzani, 2001; Edwards, 2009; Font, 2007; Font & Acevedo, 2003; Lakoff & Núñez, 2000; Núñez, 2000, 2007; Núñez, Edwards & Matos, 1999; Presmeg, 1992, 1997, 2005; Sfard, 1994, 1997, 2000; Pimm, 1981, 1987; Robutti, 2006) have pointed to the role of metaphors in the teaching and learning of mathematics, and some of them have reflected upon embodied cognition theory. Sriraman and English (2005), in their survey of theoretical frameworks that have been used in mathematics education research, talk about the importance of embodied cognition theory.

Since there are disputes within mathematics education research about the nature of embodiment in mathematics, it will be useful to begin by considering the constructs used in this research and the way in which they are understood.

Our view that people generate image schemas positions us, in accordance with Johnson (1987), at an intermediate point between two opposing perspectives regarding how information is stored in long-term memory. Some psychologists argue that long-term memory contains spatial images, whereas others refuse to accept that such mental images are stored in memory in a figurative format other than the propositional one. The proposal of Johnson (1987), whose origins can be found in Kant's theory of imagination, consists precisely in postulating certain schemas, known as image schemas, which are situated halfway between images and propositional schemas. These schemas are constituted by multiple bodily experiences undergone by the subject. Some of these experiences share characteristics that are incorporated within the image schema. Both the experiences and the shared characteristics are a consequence of situations that have been physically and repeatedly lived. Briefly, an image schema is a condensed redescription of perceptual experience for the purpose of mapping specific structure onto conceptual structure[2]. For example, Fig. 1 represents the "container" image schema.

Image schemas help to build reasoning by means of conceptual inference projections, among them metaphorical ones. Following Lakoff and Johnson (1980) and Lakoff and Núñez (2000), we consider that the metaphorical projection is the main cognitive mechanism that enables abstract mathematical entities to be structured by means of bodily experiences. We interpret 'metaphor' as the comprehension of an object, thing, or domain in terms of another one. Metaphors create a conceptual relationship between an initial or source domain (the one we are familiar with) and a final or target domain (the new or abstract one), while properties are projected from the first to the second domain. Because metaphors link different senses, they are essential for people in terms of constructing meanings for mathematical entities: "(…) a large number of the most basic, as well as the

[2] See Oakley (2007) for a presentation of its terminological history, the review of a range of studies illustrating the application of image schemas, as well as the studies' review that establish the psychological and neuropsychological reality of image schemas.

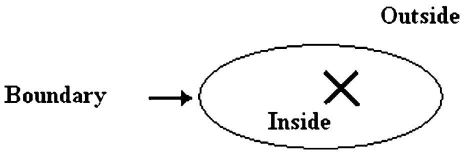

Fig. 1 The "container" image schema

most sophisticated, mathematical ideas are metaphorical in nature" (Lakoff & Núñez, 2000, p. 364).

In relation to mathematics, Lakoff & Núñez distinguish two types of conceptual metaphors:

- Grounding metaphors: these relate a target domain within mathematics to a source domain outside it.
- Linking metaphors: these maintain the source and target domains within mathematics and exchange properties among different mathematical fields.

For example, the "container" image schema may be metaphorically projected in order to understand parts of other more abstract domains, such as that given by the theory of classes. In this case, we have the grounding metaphor "classes are containers".

We are also aware that only some aspects of the source domain are revealed by a metaphor and, in general, we do not know from which aspects of the source domain the students map. Let us consider the metaphor 'mathematics is a building'. Here, the desired inferences would be that mathematics is a subject that one should learn from basic facts in order to have a strong support for learning the next subject. Thus what is being mapped is that a building has strong foundations in order to place one floor over the other; however, someone could look for the windows and doors and get a totally different perspective on it.

In our view, the unitary-systemic duality has a central role in the work of Lakoff and Núñez (2000). On the one hand, a metaphor is unitary (A is B). Yet on the other, a metaphor allows us to generate a new system of practices (systemic perspective) as a result of our understanding the target domain in terms of the source domain. Lakoff & Núñez develop the elementary-systemic duality for different metaphors. A good example is the metaphor of the container (Núñez, 2000, p. 13; Table 1).

Table 1 The metaphor "classes are containers"

Classes are containers[a]		
Source domain	→	Target domain
Container schemas	→	Classes
Interiors of container schemas	→	Classes
Objects in interiors	→	Class members
Being an object in an interior	→	The membership relation
An interior of one container schema within a larger one	→	A subclass in a larger class
The overlap of the interiors of two container schemas	→	The Intersection of two classes
The totality of the interiors of two container schemas	→	The union of two classes
The exterior of a container schema	→	The complement of a class

Systemic perspective

[a] Unitary perspective

In fact, most research on metaphors has focused predominantly on such a duality. In other words, given a metaphor, the source and the target domains are decomposed to determine which concepts, properties and relationships, etc. from the source domain are transferred to the target domain. The systemic view of a metaphor leads us to understand it as a generator of new practices.

A conceptual metaphor is directly related to the person who creates it and is manifested in his/her discourse, gestures, or written documents. In line with Font, Godino, Planas and Acevedo (2010), we consider it necessary to make a distinction between metaphorical expressions and conceptual metaphors, regarding them as highly interrelated but different ideas.

> Conceptual metaphors enable metaphorical expressions to be grouped together. A metaphorical expression, on the other hand, is a particular case of a conceptual metaphor. For example, the conceptual metaphor "the graph is a path" appears in classroom discourse through expressions such as "the function *passes* through the coordinate origin" or "if *before* point M the function is ascending and *after* it is descending then we have a maximum". The teacher is unlikely to say to students that "the graph is a path" but, rather, will use metaphorical expressions that suggest this. (p.19)

Figure 2 shows the process followed to characterize the grounding metaphors considered in the present research. The process, which is more parallel than sequential, begins by identifying verbal expressions that, in our view, suggest a conceptual metaphor, and those which suggest the same metaphor are then grouped together. The next step involves considering the image schema that could be the source domain and the experiences that might generate it. The source and target domains are then decomposed to determine which concepts, properties, and relationships, etc. from the source domain are transferred to the target domain. This process takes into account a series of image schemas that have already been characterized in other studies (container, whole-part, etc.).

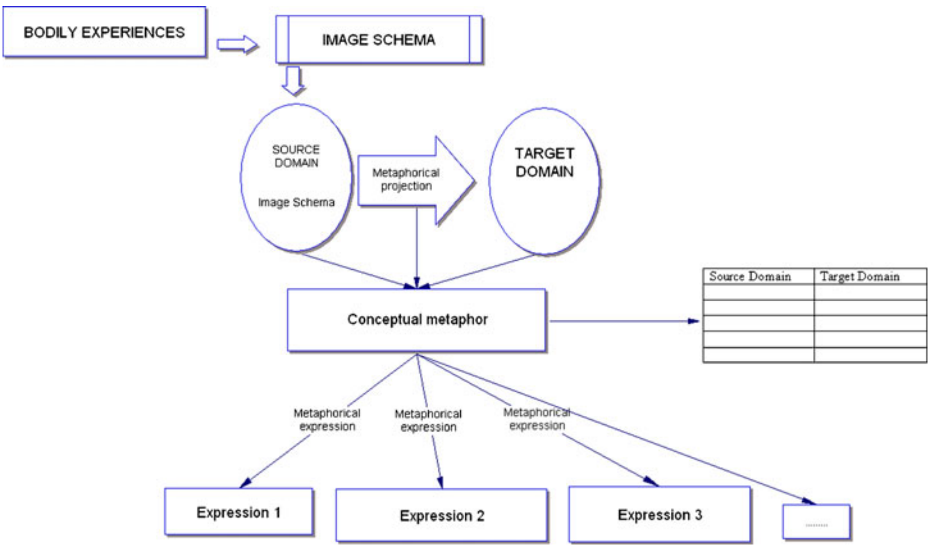

Fig. 2 Characterization of the grounding metaphors considered in this research

Previous research on both motion and graphing (Acevedo, 2008; Bazzani, 2001; Font, 2007; Font & Acevedo, 2003; Lakoff & Núñez, 2000; Malaspina & Font, 2009; Nemirovsky, Tierney & Wright, 1998; Núñez, 2007; Radford, 2009; Radford, Miranda & Guzmán, 2008; Robutti, 2006) highlights the role of the body in mathematical thinking with respect to reading and elaborating Cartesian graphs for functions. Some of these studies (Acevedo, 2008; Lakoff & Núñez, 2000; Núñez, 2007; Font & Acevedo, 2003; Malaspina & Font, 2009) have stressed the importance of the "Source-Path-Goal schema" in understanding the graphs of functions (Fig. 3). This schema has the following elements: a trajector (moving object), a source (origin or starting point), a goal (destination or endpoint), and a path (series of contiguous locations from source to goal).

Lakoff and Núñez (2000) consider that the graphs of functions are structured through the metaphorical mapping of this schema. Such a mapping conceptualizes the graph of the function in terms of motion along a path. Acevedo (2008) describes how this metaphorical mapping forms part of the dynamic processes of teaching and learning of Cartesian graphs for functions at the high school level. Malaspina and Font (2010) consider that the "optimizing schema" can be considered as being derived from the "Source-Path-Goal schema", e.g., when a function is described as "going up", "reaching" a maximum, and "going down" again. In this way, the idea of the ups and downs of a road is essential to the understanding of the concepts of maximum and minimum.

The work described in the present paper confirms the importance of the "Source-Path-Goal schema" for understanding the graphs of functions. However, it is shown that one must consider other metaphorical projections that are also present in the dynamic processes of teaching and learning to read and elaborate Cartesian graphs for functions at high school level.

3 Methodology

This research presents an analysis of teachers' discourse while teaching the graphic representation of functions in Spanish high schools. Our focus was on teachers' discourse and practice, so the students' discourse and practice[3] only appear when interacting with the teacher.

The teachers who participated in this research did so voluntarily and gave their specific consent to our interference within their teaching work (class observations, video recording, analysis of working materials, etc.).

All the participating teachers were men (since the female teachers we contacted preferred not to take part). For each teacher, we analyzed the textbook units referring to functions, as well as his own material (work sheets, examinations, etc.). Teachers A, B, C, D, and W were video recorded while giving a 1-h class in which they explained graph functions to their students. The other two, less-experienced teachers (E and F) were asked to explain to two research interviewers how they would explain the graph of the function $f(x) = \frac{x^2-5x+4}{x-5}$ to their students. The purpose of this change of format was to observe whether there was a significant reduction in the presence of metaphors in their explanation for two reasons: (1)

[3] For us, to learn mathematics is to become able to carry out a practice and, above all, to perform a discursive reflection about it that would be recognized as mathematical by expert interlocutors. From this perspective, we see a teacher's speech as a component of his professional practice. The objective of this practice is to generate not only a type of practice within the student, but also a discursive reflection about what can be considered as mathematics.

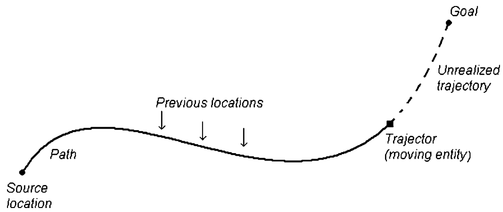

Fig. 3 The "Source-Path-Goal schema" image schema

their limited teaching experience; and (2) the fact that although their audience (their students) was present in theory, they were not actually in a classroom situation.

All the teachers were subsequently interviewed about the use of metaphors in their discourse. This was an open-ended interview in that the questions put did not require a closed answer. In terms of the degree of structure, the interview procedure was mixed. Beginning with an 'interview script' the teachers were first made aware that they used certain expressions and gestures. They were then asked why they did this and what consequences they thought these expressions and gestures could have in terms of their students' understanding.

All the teachers, shown in the table below, had a degree in mathematics or physics. Teachers D and W worked in secondary schools in the center of Barcelona, teachers E, F, and W taught in a school located in what could be called the first industrial belt around Barcelona, and professor W was the tutor of teacher E. Teacher C taught in a small town in the province of Barcelona, while teacher A worked in a medium-sized town in the province of Tarragona, which borders that of Barcelona (Table 2).

In order to analyze teaching processes effectively, we videotaped teachers' lessons and transcribed them. These transcriptions were organized into three columns: (1) transcriptions of the teacher's and students' oral discourse, (2) teachers' blackboard notes (including drawings), and (3) comments on the teacher's gestures (as shown on the tape).

Table 2 Characteristics of teachers who participated in this research

Teacher	Institution	Municipality	Level	Teacher degree	Years in service	Time of Class
A	IES Baix Penedès	El Vendrell	Senior secondary school teacher	Mathematics	15	Evening
B	IES XXV Olimpiada	Barcelona	Professor of secondary education	Physics	30	Day
C	IES Vilatzara	Vilatzara	Professor of secondary education	Physics	25	Day
D	IES Les Corts	Barcelona	Professor of secondary education	Mathematics	30	Day
E	IES Margarida Xirgu	L'Hospitalet de Llobregat	Trainee teacher on placement	Mathematics	No experience	Day
F	IES Margarida Xirgu	L'Hospitalet de Llobregat	Temporary teacher	Mathematics	7	Day
W*	IES Margarida Xirgu	L'Hospitalet de Llobregat	Professor of secondary education	Mathematics	26	Day

It was necessary to separate the transcriptions into units of analysis. One possible way of doing this was to take the construct "didactic configuration" as the basic unit of analysis. Godino, Contreras and Font (2006) consider that a *didactic configuration*—hereinafter referred to as a DC—is established by teacher-student interactions based around one mathematical task.

The teaching process with mathematical content takes place in a timeframe that includes a sequence of didactic configurations. Although the basic criterion for determining a DC is the performance of a task, grouping into didactic configurations is flexible and at the researcher's discretion. Division of the classroom session into didactic configurations enables subsequent macroscopic analysis of a wide range of didactic configurations, while finer (microscopic) analysis will mainly be carried out on a much smaller number of these didactic configurations. In our research, once a DC had been defined we focused our analysis on the phenomena related to the use of metaphors seen within it.

The procedure followed in order to determine the metaphors used by teachers is that set out in Fig. 2 (in Section 3): (1) identify metaphorical expressions; (2) group together those that suggest the same metaphor; (3) consider the image schema that could be the source domain; and (4) decompose the source and target domains in order to determine which concepts, properties, and relationships, etc. from the source domain are transferred to the target domain. Following McNeill (1992), we consider that gesture and language are one system, and that it is this system which produces the conceptual metaphor; hence, gestures were also taken into account. However, it should be noted that the study of gestures was a secondary objective in the present research, and they were only listed in a column of the transcriptions when it was considered that they reinforced the effect of a given metaphorical expression.

4 Data analysis

In this section, data will be analyzed according to the two research questions stated previously.

4.1 Regarding question one

Concerning the first question (What types of metaphors does the teacher use to explain the graphic representation of functions at the high school level?), we found that teachers structured their discourse according to certain types of metaphors. The image schemas used, which will be explained in the following sections, were the orientation image schema, source-path-goal image schema (path image schema), container image schema, part-whole schema, and object image schema. The related conceptual metaphors found in teachers' discourse will be shown.

4.1.1 Fossilized metaphors

In our view, a fossilized metaphor is a type that the academy (institution) considers as a literal expression, while remaining unaware of its metaphorical origin. This metaphorical origin can be observed from the symbols utilized, e.g., the arrow (→) used in limit notations, as well as from the way that this symbol is read (limit of $f(x)$ when x goes toward positive infinity). Other very common expressions in lessons, such as "x goes till" or "limit from the right", are considered by teachers as literal expressions. This discourse can be illustrated through the following fragment of a lesson (Table 3).

This teacher could be seen to emphasize fossilized metaphors, not only through his oral and written explanations but also in his gestures. However, he also used other types of

Table 3 Review of limit definitions (Teacher A)

Transcription	Blackboard
T: There may be many ways… and in any of them what we are studying is exactly this: the *limit of f(x) when x approaches positive infinity.* What is the limit of this function?	$\lim f(x)$ $x \to +\infty$

metaphors in his discourse, and here we will focus on ontological, orientational and other varieties of expression.

4.1.2 Ontological metaphors

The group of grounding metaphors includes the ontological type, where we find the object metaphor. The object metaphor is a conceptual metaphor that has its origins in our experiences with physical objects and enables the interpretation of events, activities, emotions, and ideas, etc. as if they were real entities with properties. This type of metaphor is combined with other ontological, classical metaphors such as that of the "container" and that of the "part-whole". The combination of these types leads to the interpretation of ideas and concepts, etc., as entities that are part of other entities and which are constituted by them. This interpretation is clear in these axioms, as they are mentioned in a classical Spanish textbook on geometry (Puig, 1965, p. 4):

Ax. 1.1. We recognize the existence of infinite entities called <points> whose set will be called <space>.

Ax. 1.2. The points of the space are considered to be grouped in partial sets of infinite points called <planes> and those from each plane into other partial sets of infinite points called <lines>.

Ontological metaphors can also be found in the discourse of teachers when they explain the graphical representation of functions. According to Lakoff and Johnson (1980, p. 25), this type of metaphor "allows us to pick up parts of our experience and treat them as discrete entities or substances of a uniform kind".

In this research, ontological metaphors were considered as a group of metaphors (container, whole-part, object metaphor, etc.) that result from the projection of image schemas which, in our view, share a 'common territory'; therefore, there may be a certain hierarchy among them.

The following sub-sections present a more detailed analysis of three conceptual metaphors of the ontological kind which were found in teachers' discourse (or in the text book they used): object metaphor, container, and whole-part. In this case, no hierarchy is established between them.

4.1.3 Object metaphor

The object metaphor is a grounding metaphor that maps the object image schema in mathematics. This image schema is experientially grounded in our physical and social interactions with our own bodies and with other discrete entities in the world. The object image schema is not described in detail in the seminal works by Johnson (1987) and Lakoff (1987), although the former does include it in his inventory of image schemas (Johnson 1987, p. 126). The projection of this image schema enables us to speak and reason about different kinds of abstract entities as if they were discrete objects. For example, in teachers' discourse, mathematical entities are presented as "objects with properties" that can be physically represented (on the board, with gestures, etc.).

In the case of graphs of functions, students hear or read metaphorical expressions such as "this function always exists", "find the function", "find the domain and the cut points", or "you mustn't confuse the function with its graphical representation", etc. These expressions suggest the grounding ontological metaphor "mathematical objects are physical objects" which, in the present article, we shall call an "object metaphor". Table 4 refers to the source and target domains that intervene in the interpretation of this metaphor.

Here, metaphorical expressions of the object metaphor occur when the mathematics teacher refers to the graph of a function as an object with physical properties. The teachers use verbal expressions and gestures that suggest the possibility of manipulating mathematical objects as if they were objects with a physical existence:

Teacher C: No, not the derivative of the numerator! You multiply by the denominator as it is, minus the numerator multiplied by the derivative of the denominator. OK. Now you divide it by the denominator... squared (...) This is the first derivative. *Now, what's next? To operate, to manipulate*... What's left?

Teacher B: (…) you know how *to find* maximums and minimums… before the holidays we worked on problems involving maximums and minimums. So, Rocío, what do you think you have to do *to find* maximums and minimums? [They are representing graphically the function $f(x) = x^3 - 6x^2 + 9x + 5$.]

Teacher B: Can we *do* anything in the equation before applying the formula? [They are looking for the abscissa values that cancel out the derived function of $f(x) = x^3 - 6x^2 + 9x + 5$ and the teacher is expecting the students to say that 3 can be extracted as a common factor in the following equation: $3x^2 - 12x + 9 = 0$.]

Teacher B: Now, *where would you put* the maximum and the minimum? [They are representing graphically the function $f(x) = x^3 - 6x^2 + 9x + 5$.]

The distinction between the representation and the function object was not made explicit by teachers in their discourse, yet they repeatedly made use of two different representations of the function object (the symbolic expression and the graph), as well as representations of non-ostensive objects that do not exist (for example, when the teacher says that $f'(a)$ does not exist because the graph of $f(x)$ has a pointed form at $x=a$). The use of these different representations (Duval, 1995, 2006), along with representations of non-ostensive objects that do not exist (Font et al. 2010), fosters the belief that mathematical representations are different from the mathematical objects they represent.

Table 4 Mathematical objects are physical objects

Metaphor Projection	
Source domain: object image schema	Target domain: mathematics
Physical object	Mathematical object
Physical objects are manipulated, found, discovered, etc.	Mathematical objects are manipulated, found, discovered, etc.
Physical objects are different from their material representations (for example, a clock is different from the drawing of a clock)	Mathematical representations are different from the mathematical objects they represent
Properties of the physical object	Properties of the mathematical object
Physical objects exist	Mathematical objects exist

In both the textbook used by the teacher and in his own discourse, functions and their graphs are presented as objects with properties (for example, if the derivative is positive in $x=a$, then the function is growing at $x=a$).

One of the properties of physical objects that is normally not questioned is that they exist independently of people (although they can be constructed by them). Somehow the type of existence of objects such as chairs, trees, and stones, etc. is translated into the world of mathematical objects, thus making it possible for the latter to be likewise regarded as existing independently of people.

Although the form of existence referred to above is the most problematic (and in explaining it, one can choose between realist accounts, such as the Platonic or empiricist, and constructivist accounts, among others), there is another sense in which mathematical objects can be said to exist and which does not prove problematic for the majority of mathematicians (with the exception of the intuitionists). In order to distinguish between these two types of existence, and following Carnap (1950)[4], we consider that the existence referred to in the previous paragraph would be an external existence, whereas when mathematicians state that a given mathematical object exists (or not) they do so from the perspective of internal existence.

When teachers talk about 'existence' they normally do so from the internal perspective, although at times they do adopt the external viewpoint. For example, in this classroom discussion (first on the domain of the logarithm function, and later on the domain of the square root function), and while teaching the graphical representation of functions[5], "existence" is considered within the language game (Wittgenstein, 1953) of mathematical discourse (internal).

Teacher A: The domain goes from zero to infinity because logarithms of *negative numbers do not exist*, the logarithm *of minus one does not exist*. Should the zero be included?

Teacher A: Not the negative… because *the square root of a negative number does not exist.* We could also say the same real numbers but without the negatives, or even easier, all the positive numbers… we can write it like this, with an interval, from zero to infinity… now the zero is included.

In contrast, in the following paragraph, in which teacher W explains the graphical representation of functions to students, he explicitly mentions the idea of existence, although he does so in a rather controversial way:

Teacher W: So... this function always *exists*, the domain will be all real numbers and there *won't be* any vertical asymptote [the function is $f(x) = x^3 - 3x^2 - 45x + 2$.]

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[4] The term existence can be understood from an absolute or a relative perspective, as Carnap discussed in *Empiricism, semantics and ontology* (Carnap, 1950). In relation to the existence of abstract entities, he considered "internal" and "external" questions. "If someone wishes to speak in his language about a new kind of entity, he has to introduce a system of new ways of speaking, subject to new rules; we shall call this procedure the construction of a linguistic *framework* for the new entities in question. Now we must distinguish two kinds of questions of existence: first, questions of the existence of certain entities of the new kind *within the framework*, which we can call *internal questions*; and second, questions concerning the existence or reality *of the system of entities as a whole*, called *external questions*. Internal questions and possible answers to them are formulated with the help of the new forms of expressions. The answers may be found either by purely logical methods or by empirical methods, depending upon whether the framework is a logical or a factual one. An external question has a problematic nature which is in need of closer examination" (Carnap, 1950, p. 20). As Carnap said, the internal existence within a particular linguistic framework is not problematic, in comparison to the external existence of the system of entities as a whole.

[5] At this school level, in our country, students are not taught about complex numbers.

In line with Font et al. (2010, p. 16), we observe a deviation here in the “expected” use of the word “exists” within the language game of the mathematics discourse. It would be reasonable to affirm that the images of the values in the domain ‘exist’ or are defined. However, when attributing existence to the whole function instead of talking about its images, the teacher is making a use of the word “exists” that can lead to the function being understood as a “real” object with properties, like a chair or a person. Moreover, by doing so, the teacher can promote movement from the mathematical internal existence of the object to a physical external existence.

4.1.4 Container

The conceptual metaphor “a domain is a container of points” is also a grounding metaphor (Table 5), where the source domain is the image schema of container (see Section 2). In the case of graphs of functions, students hear or read metaphorical expressions such as “Is this value included in the domain?” which suggest this grounding ontological metaphor.

This can be exemplified through an explanation given by teacher A when he seeks to explain what the domain of the function $f(x)=\ln x$ is, as well as that of the function $f(x) = \sqrt{x}$.

Teacher A: Yes, from zero to positive infinity… this is the domain, because there is no logarithm of negative numbers, there is no logarithm of minus one. *Is the zero included or not?*

Teacher A: (…) we can express it more easily… we can put it in interval form, from zero to positive infinity... *now the zero is included, this time it is…*

The image schema of the container is also projected when a graph is considered to be contained within the Cartesian plane.

4.1.5 Part-whole

The conceptual metaphor “points are part of a graph” is another ontological metaphor (Table 6). The source domain is the part-whole image schema. This schema (Lakoff, 1987, p. 273-274) is experientially grounded in our perception of our bodies as wholes made up of parts and other objects with parts. In the case of graphs of functions, students hear or read metaphorical expressions such as “this function is made up of parts”, etc., which suggest this grounding ontological metaphor.

Table 5 A domain is a container of points

Metaphor projection	
Source domain Container image schema	Target domain Domain of functions
Container	Domain
Object inside the container	Point of the domain
To be an object in the container	To be included in/To be in (the domain)
A container inside another container (the second container is bigger than the first)	A part of a domain that is part of a bigger domain
The exterior of a container	Points that do not belong to (are not part of) the domain

This metaphorical projection is of a grounding type and can be exemplified with an explanation given by two different teachers:

Teacher A: For example, this function $f(x) = 1/(x+1)\ldots$ the domain of this function *is built by the set of numbers*… so when I substitute x by these numbers I can perform the whole of this calculation… in other words, I can find the image.

Teacher B: There are no parabolas here… *what are the other interesting points?* I mean, I found this one and this one… they are the points that cut the axes. *What are the other points?* [They are representing graphically the function $f(x) = x^3 - 6x^2 + 9x + 5$ and the teacher is hoping the students will reply by saying that the other points are the maxima and minima.]

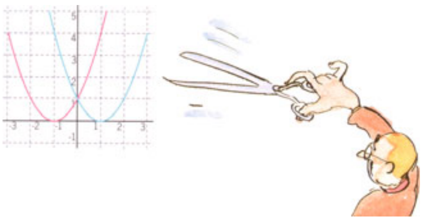
A good example for which the part-whole image schema is relevant is when a function is defined as a solution of a differential equation, that is, as a particular element of a set of solutions. A metaphorical expression of this projection is the name given to this type of function: functions defined in parts.

Other metaphorical expressions of this conceptual metaphor (which we have underlined) can be found in the text book used by Teacher *A* when introducing this type of function (Bujosa, Cañadilla, Fargas & Font, 2003, pp. 99-100):

Piecewise function

In the activities proposed below you will work with types of function graphs that are made up of parts of other graphs.

Activity 32: In the same coordinate axes we have superimposed the graphs of the functions $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.



a) Match each graph to its formula:

Figure 4

b) If we now look at just one graph, find the images of -1, 0 and 1. Does this graph correspond to a function?

Activity 33: We have now modified the previous graph and it looks like this:

a) Look at the graph and find the images of the whole numbers that appear in it.

b) Is it the graph of a function? Why?

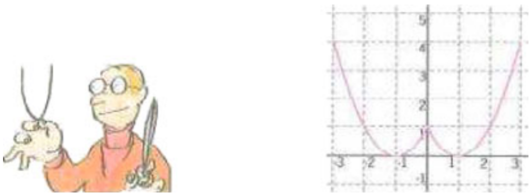


Figure 5

This final graph is made up of parts of the graphs of the functions $f(x)$ and $g(x)$. What is the formula of the function whose graph is that of the previous activity? This function must have a <<part>> of $f(x)$ and another <<part>> of $g(x)$. Its formula is:

$$h(x) = \begin{cases} x^2 + 2x + 1 \text{ if } x \leq 0 \\ x^2 - 2x + 1 \text{ if } x > 0 \end{cases}$$

(…) Functions defined from other functions – as we have seen in the previous example – are called *piecewise functions*.

Table 6 Points are part of a graph

Metaphor projection	
Source domain	Target domain
Part-whole image schema	Graph of function
Whole	Graph
Constituent parts	Points of the graph
The whole is more than the part	The observed graph is a part of the whole graph (e.g., when the domain is *R*)
The union of all the parts is the whole	The graph is made up of all the points

With regard to this fragment, it should be emphasized that the metaphorical projection is suggested by a combination of verbal metaphorical expressions and gestures (represented by the drawn figure).

The metaphor which considers that graphs are made up of parts is implicitly presented in computer software that visualizes the graph ‘all at once’ based on a table of values, for example, an Excel spreadsheet (Acevedo, 2008, pp. 101-102), or when one has to answer questions about the behavior of the function for abscissa values that are beyond the observable interval.

4.1.6 Orientational metaphors

Orientational metaphors do not structure one concept in terms of another, but rather organize entire systems of concepts with respect to each other. The name ‘orientational metaphor’ is connected with the fact that, in most cases, such metaphors have something to do with spatial orientation, including image schemas such as central-peripheral, in-out, front-back, up-down, among others (Lakoff & Johnson, 1980, p. 14). From this perspective, orientational metaphors enable us to organize the system of mathematical objects that are involved in the graphical representation of functions in terms of the orientation system we use to organize our experiences in the physical world; thus, we can describe the position of the objects around the subject, although always with respect to a reference point.

Table 7 Orientational metaphor

Metaphor projection	
Source domain	Target domain
Up	Values of y such as $y>0$
Down	Values of y such as $y<0$
Right	Values of x such as $x>0$
Left	Values of x such as $x<0$
Something is on the left side of another thing that is on the right side	$x_1<x_2$
Something that is below/less than another thing	$y_1<y_2$
Something is between two things	$x_1<x_2<x_3$; $y_1<y_2<y_3$

Orientational metaphors include at least the following cross-domain correspondences (projection), as shown in the Table 7.

In the case of the graphs of functions, students hear or read metaphorical expressions such as "a solution is between…" or "the minus 1 would be here below", etc.

This metaphorical projection can be exemplified with an explanation given by Teacher B, in which he uses Bolzano's theorem to find the abscissa of the cut point for the function $f(x) = x^3 - 6x^2 + 9x + 5$ with the abscissa axis:

Teacher B: (…) so, where would it be… where would the minus 1 be? The minus 1 would be here below…

[Although he doesn't say so, he is referring to the point on the abscissa $x=-1$.]

[His body position when drawing the image of $x=-1$ is inclined slightly downwards.]

Teacher B: Zero point one… could you give another part using a criterion, one that is easier to do? It's between minus 1 and zero… now, we would want it where?

Student: In the middle.

Teacher B: Which value is in the middle?

Student: Minus zero point five.

Another example is given by Teacher E when commenting on the result of studying the sign of the function $f(x) = \frac{x^2-5x+4}{x-5}$:

Teacher E: (…) so, we're left with this region and this region here at the top… OK?

Other expressions used by the teachers and which, in our opinion, suggest orientational metaphors are those which contain the words "horizontal" and "vertical". In a vertical plane, there is only one horizontal direction and one vertical direction perpendicular to it. The metaphor can then be justified and understood: when laying the work sheet on a table (as is usually the case for students' material) these two directions become the conventional "horizontal" and "vertical" directions. It is worth emphasizing that the teachers almost always identify the ordinate axis as the vertical axis and the abscissa axis as the horizontal axis, even though the textbook never makes this identification.

Teacher A: (…) in $x=0$ it shows a minimum and the derivative in $x=$o is zero, as we would expect, because now the tangent line is horizontal… [While he says this he gestures with his hands, indicating the horizontal position of the tangent line on the graph on the blackboard.]

4.1.7 Dynamic metaphor—fictive motion

We observed that the teacher's discourse suggests to students that they should understand the graph as a path that one walks along or a line which one follows. Let us consider an example from Teacher A (Acevedo, 2008, pp. 139-140).

Teacher A: What we have to do is create a variation table. *If before zero it is increasing… if after zero it is decreasing… If before and after zero it is increasing, then there is an inflexion point. If before zero it is increasing and after zero it is decreasing, then there is a maximum. If before zero it is decreasing and after zero it is increasing, then there is a minimum.* [He explains what they have to do when the value of the second derivative is also zero.]

Table 8 Graph is a path

Metaphor projection	
Source domain	Target domain
Path schema	Graph of functions
Path	Graph
A location on a path	A point on the graph
To be on a path	A point that belongs to the graph.
The origin of the path	Origin of the graph
The end of the path	The end of the graph (e.g., positive infinity)
To be off the path	A point that does not belong to the graph

According to Talmy (2000), these are typical examples of "fictive motion", and are grounded in the image schema of a "Source-Path-Goal". In some cases, the start point is "minus infinity" and the end point is "plus infinity".

We can add that "the graph is a path" is a grounding metaphor, and the source domain is the image schema Source-Path-Goal (Table 8).

The metaphor "the graph is a path" along which the points move is favored by the fact that the graph, in turn, is considered as a path that passes through different regions.

Teacher F: (…) this shows us that the graph is going to pass through here and here. [He comments on the result of studying the sign of the function $f(x) = \frac{x^2-5x+4}{x-5}$.]

Table 8 shows some of the interesting difficulties faced when attempting to create metaphors (or use image schemas) more accurately. Many graphs do not have an "origin" or an "end"—the end of a path cannot be mapped to a non-location-like positive infinity, at least not without taking into account Núñez's idea of "completed" infinity (Núñez, 2005). However, talking about a section of a graph does allow this image schema to be used.

We observed two kinds of situation with respect to the start and end points of the graph at infinity. One is when the teacher draws the graph, beginning at minus infinity and ending at plus infinity, but uses gestures to suggest that both infinities are off the drawn graph, i.e., gestures which indicate that the graph could continue. Another is when the graph has already been drawn and the teacher has to make gestures involving the graph in order to clarify some point to the students, and in this case he is less careful. For example, when Teacher B talks about the continuity of the function he situates himself at the start of the graph, where the minus infinity sign is, and runs the chalk along the graph until he reaches its end-point, which is indicated by the plus infinity sign. However, he makes no attempt to highlight through gestures that the drawn graph is only a part of the graph of the function.

Teacher B: So, Marina, if the function is continuous, then this is very important… it means that I can always keep drawing without lifting my pencil from the paper or the chalk from the board… isn't that right?

4.1.8 Interaction of metaphors

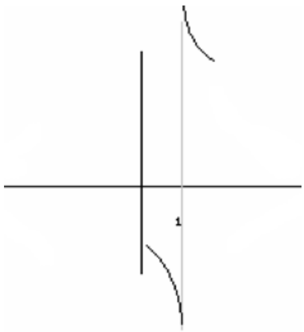
The teachers' discourse sometimes reveals the combined and flexible use of metaphorical expressions (often accompanied by gestures) which suggest, on the one hand, static metaphors (whose source domain is the image schemas of the container and the part-whole) and, on the other, metaphors such as the graph is a path (whose source domain is the Source-Path-Goal image schema) or orientational metaphors. This is illustrated by the following fragment involving Teacher B:

Teacher B: (…) applying Bolzano's theorem, which is very easy… if you have one value, minus 1, the image is negative, and with another value the image is positive… if the function has to go from here to here, then there must be at least one point that cuts it… in other words, there is at least one solution and it is between this value and this one… the more you limit this the closer you'll get to the possible solution, but the only possibility is this.

Another example can be seen in this fragment involving Teacher A, who is explaining how to interpret graphically $\lim_{x \to 1^-} \frac{1}{x^2-1} = -\infty$ and $\lim_{x \to 1^+} \frac{1}{x^2-1} = +\infty$, in which there is a combination of expressions, some suggesting the metaphor that the graph is a path, while others suggest orientational metaphors (Table 9).

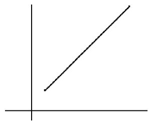
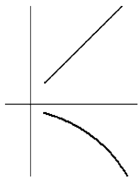
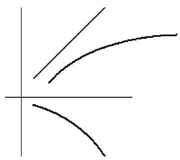
Another combination of metaphorical expressions is that which results from combining the object metaphor with the metaphor that the graph is a path. An example can be found in the following fragment involving Teacher A (Table 10).

Table 9 Combination of metaphorical expressions which results from combining the object metaphpr with the metapor that the graph is a path

TRANSCRIPTION	BLACKBOARD Figure 6	OBSERVATIONS
Teacher A: Remember that there is not only one notation… what can we say graphically about it? Well, when one *approaches from the left* … like this… the graph *tends toward negative infinity*, and increasingly approaches this line… and when *x approaches 1 from the right* the graph *goes up*, increasingly approaching this vertical asymptote or this line, so we say that this line *x equals 1 is a vertical asymptote.*	The straight line $x = 1$ is a vertical asymptote 	He moves his hands to indicate the trend of the graph when x tends toward 1.

The writing on the board, blended with his movements, produced an animation of static mathematical objects. We could also say that it produces a personification of those objects because the objects started to do things, to grow, and so on. The object schema enables students to understand that mathematical entities are objects, just as people and animals are

Table 10 Asymptotes and behavior at infinity (Teacher A)

TRANSCRIPTION	BLACKBOARD Figures 15-17	OBSERVATIONS
80 : One thing we have learned about the graphical representation of a function is its behavior at infinity. *What do we do when x tends toward infinity? What does the graph of the function do when x tends toward infinity?* It could go like this, going *toward positive infinity.* 81 : *It could go like this, going toward negative infinity.* 82 *P: It could also grow and stabilize at a number, like this.*	  	The teacher moves his hands, making movements as if he were continuing to draw the segment in the graph, suggesting that it continues indefinitely.

and all living beings. In addition, expressions such as "What does the graph do when *x* tends toward infinity?" facilitate the notion of mathematical objects as dynamic objects.

4.2 Regarding question two

Are teachers aware of their use of metaphors in their discourse? If so, how do they control this use of metaphors?

In order to evaluate this aspect, we recorded classes given by teachers *A*, *B*, *C*, and *D* in their usual workplace, and followed this up with a semi-structured interview. Teachers *E* and *F* were asked to tell the two interviewers how they explained to students the graphical representation of a function, and this was also followed up by a semi-structured interview. The transcriptions of their explanations regarding the graphical representation of functions reveal, among other things, the following significant aspects: (1) teachers use different metaphors in their discourse (orientational metaphors, the asymptote as a barrier, the animation of mathematical objects, etc.); (2) they use a dynamic discourse accompanied by hand gestures in order to explain the static components of the graph. In their discourse, the metaphor "the graph of a function can be considered as the trace left by a point that moves over the graph" is present in statements such as "it comes from here" or "it passes through…”, etc.; and (3) there is no significant reduction in the use of metaphors in the discourse of teachers E and F, despite the fact they have much less professional experience than do the other teachers.

Interviewing the four professors we observed that:

1. They were not aware of using metaphors.
2. After being “forced” to observe the metaphors, they used in the video, they were amazed by the metaphors they used.
3. There was no control over the metaphors while teachers were unaware of using them. Their discourse did not include any mention of inferences that could be made by students but which were not mathematically accepted.
4. For all the teachers, the use of metaphors was seen as a way of facilitating students’ understanding of this topic.
5. When we asked the teachers about the possible disadvantages of using their chosen metaphorical expressions, four of them said there was no problem or no risk involved. However, Teacher E said that if the use of metaphors was not complemented by rigorous explanation, it could have a damaging effect on students’ understanding.

The table below shows the transcript from parts of the interviews (Table 11).

This interview fragment shows how Teacher *C* is unaware of his use of dynamic metaphors and, therefore, he does not control them. Following the interviewer’s questions, Teacher C realizes that he makes use of these metaphors, but still maintains that their usage facilitates students’ understanding, giving little consideration to the difficulties which could actually arise. In fact, he believes that the use of metaphors does not lead to any conceptual errors among his students.

5 Final considerations

This paper has shown that conceptual metaphors are relevant tools for analyzing teachers’ mathematical discourse in the classroom. The present study provides empirical data that contribute to a better understanding of how metaphors are used when teaching the graphs of functions to high school students. Although this usage is inevitable and sometimes unconscious, it is of fundamental importance in building and talking about mathematical objects.

Teachers structure their discourse about graphs of functions around a small group of metaphors, which are projections of the following image schemas: the orientational schema, the path schema, the container schema, the part-whole schema, and the object schema.

We have argued that the projection of the object schema plays a central role in the classroom discourse in which teachers talk about mathematical objects. It has also been stated that the object schema shares a common territory with the container schema and the part-whole schema, without any hierarchy being established among them. One question that remains open (Santibáñez, 2002) concerns the relationship between these image schemas. For example, one could consider that the object image schema is the fundamental schema and that the others are derived from it, or alternatively, that there is some other schema from which all (or some) of those mentioned are derived, for instance, the entity schema (Quinn, 1991) or the notion of thing (Langacker, 1987, 1998). At all events, it is not clear in this

Table 11 Fragment of the interviewing with Teacher *C*

Transcription	Observations
1. Int. In the video you use phrases like "Here it goes to..." "it passes through zero-zero", "x grows", "the numerator increases more than the denominator", and in doing so you made hand gestures. What are you trying to express by doing so?	The purpose was to determine whether the teacher was aware of using metaphors and gestures in his discourse.
2. T. *Well, I guess it's completely unconscious, because I don't think about it*. I don't tell myself to move one way or the other. *I don't know?* I think they're gestures to emphasize the ideas I am expressing. So… saying something like "it goes there"… or "it comes from here"… it should show where the function passes through and where it continues… What I mean is… imagine that you follow the function, giving values to x in an increasing way, then what would be the image or the drawing (graph) of this curve. Of course, these are things I've never really thought about, although I know I make these gestures to highlight what I'm saying. And if I didn't it would be very boring just to speak without…well, it wouldn't come over in the same way, would it?	This shows that the teacher was unaware of his use of metaphors and gestures. He assumes that the gestures emphasize an idea. In his classroom the gestures changed a static figure into a dynamic one, this being an uncontrolled metaphor about whose consequences he remains unaware.
3. Int. And have you never thought that this may lead students to make a conceptual error, while interpreting it?	This professor is unaware that his use of metaphors might hinder rather than facilitate students' understanding.
4. T. No… to a conceptual error… which one?	Although he seems briefly to reflect upon his discourse he still maintains that the metaphors facilitate students' understanding, and fails to consider the difficulties that his usage may generate.
5. Int. For instance, that a point doesn't, that a point is…	
6. T. No, the point doesn't move, and nor does the function, but… look… when you say that it passes by… it's not like a train that passes by, but rather that the line traced, well, it's as if it steps on those points. I've never actually thought about it, but I don't think… no, I've never found that this confuses students, no I don't think so	

case if there was agreement as to whether what is involved is an image schema or, rather, a construct that requires a higher level of abstraction than do the image schema.

This is an important question that is related to the ontological status of mathematical objects (or, what is the same, the mode of their existence). This is one of the problems belonging to the philosophy or mathematics (Rozov, 1989), and which is also of interest for research in mathematics education (Radford, 2008).

Our position, in accordance with the onto-semiotic approach (Godino, Batanero & Font, 2007), is that Lakoff and Nuñez's methodology of 'mathematical idea analysis' is very important in terms of explaining the emergence of mathematical objects, but that it is insufficient to describe adequately the nature of mathematical objects. This limitation was pointed out by various authors in the discussions that followed the publication of Lakoff and Núñez's book (e.g., Sinclair & Schiralli, 2003). For reasons of space, we limit ourselves here to setting out our view regarding how mathematical objects emerge from practices. In line with the onto-semiotic approach (Godino et al. 2007), we consider that the process through which mathematical objects emerge from mathematical practices is highly complex and that two kinds of emergence should be distinguished. On one level, there is the emergence of representations, definitions, propositions, procedures, problems, and arguments, while on another level there is the emergence of a mathematical object, for example, the function object, which is considered as an object that is represented by different representations and which may have several equivalent definitions, which has properties, etc. This point of view leads to the following question: Why and how are mathematical definitions and properties considered to be definitions and properties of mathematical objects that exist in a form which is different from that of their material representations? The present paper has not sought to provide an answer to this question, and we have merely stated, in line with Font et al. (2010), that the object metaphor plays a key role in the answer[6].

> The use of the object metaphor and the distinction between ostensive and non-ostensive objects described in this paper are crucial aspects when aiming to answer the following question: By what process in the classroom do students become convinced that there are mathematical objects which exist independently of people and of the language we use to describe them?

Other aspects that also play a role in generating this realist (Platonic or empiricist) conception of mathematics include: (1) the discourse that considers mathematics as an objective science; (2) the discourse about the predictive success of sciences that make use of mathematics; and (3) the process of reification and the convenience and simplicity that follows from postulating the existence of mathematical objects. (p. 19)

The use of static metaphors is already embedded in theoretical concepts, and it is therefore normal for them to be used in teachers' discourse as well. However, the use of dynamic metaphors is also present in teachers' explanations, although the teachers who participated in this study were largely unaware of how they were using them. The question that can reasonably be asked therefore is: Why does this combination of static and dynamic metaphors occur in teachers' discourse?

Two aspects should be taken into account when attempting to answer this question. The first is that in the classroom the teacher and students may take part in the same language game (Wittgenstein, 1953), because both teachers and students have the same source domain (ultimately, embodied experience), which is used to build mathematical ideas. However, teachers also have additional source domains from within mathematics, for example, the 'static' or discrete view (Núñez et al. 1999), which they can draw on but which are not immediately accessible to students.

The second aspect has to do with the fact that this discrete view is not immediately accessible to students. As well as a description in global terms, the graphical representation of functions also requires the introduction of local concepts such as increasing and decreasing at a point, etc., which must be formulated precisely in static terms, using the notion of number sets. These local concepts are very difficult for high school students to grasp, and for this reason many teachers prefer (although often without being very aware of this) to use dynamic explanations, in which the use of dynamic metaphors is fundamental. Such explanations are considered to be more intuitive.

[6] Other investigators also consider that metaphoric processes play a key role in the existence of mathematical objects. For example, Sfard (2000, p. 322) states the following: "To begin with, let me make clear that the statement on the existence of some special beings (that we call mathematical objects) implicit in all these questions is essentially metaphorical."

To conclude, we would like to point out that the focus of this article was on teacher's metaphors. However, the study here described was part of a broader research project that also considered, among other questions, the effect that these metaphors had on the students but due to space constrains we left it for another article.

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