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MATHEMATICAL OBJECTS THROUGH THE LENS OF TWO DIFFERENT THEORETICAL PERSPECTIVES: APOS AND OSA

1 Introduction

One characteristic of the research community in mathematics education is the diversity of theoretical perspectives. Strategies for finding connections among theories are needed in order to find out if there are underlying structures that make it possible to define families of theoretical positions. Recently there has been a trend where researchers have shown interest for finding some links between APOS (Action, Process, Object, Schema) Theory and semiotic theories, in particular between APOS and Onto-semiotic Approach (OSA) in order to complement analyses of results obtained in research using APOS by the use of semiotic theories (Badillo, Azcárate & Font 2011; Trigueros & Martínez-Planell 2010).

The research presented in this paper relates to the following research question: How do APOS and OSA approaches, as lenses for didactical analysis, lead to results that can be compared, contrasted and coordinated? In particular, in this paper we focus on a possible relation between OSA and APOS, taking into account how each of these theoretical approaches uses the term "object". Studying this problem is important since questions about the nature of mathematical objects, their possible types, their constitution process and the way they participate in mathematical activity are core aspects in teaching and learning mathematics. Mathematics Education theories are needed in order to reflect upon the notion of object. However, different theories, such as APOS and OSA, conceptualize this notion differently. Thus, they lead to differences in the way research problems are posed and results interpreted. As these results must be evaluated somehow, clarifying what aspects of the theoretical construct "object" are captured in these two theories and how they can be related may serve to coordinate them in a way that makes comparison and combination of results possible.

In this paper we focus on the comparison of the object construct as a way to start the "networking of theories" using three strategies: Understanding each theory and making own theories understandable; comparing and contrasting; and making a local coordination. (Bikner-Ahsbahs & Prediger 2010; Haspekian, Bikner-Ahsbahs & Artigue 2013). In order to clarify the object notion and its nature in each of the theories we proceeded as follows: 1) Starting from APOS Theory, and following its research methodology, we designed a genetic decomposition (GD) to describe the construction of a concept, derivative, to be used as context of reflection. 2) Using theoretical tools of OSA, we reflect upon different constructions of objects in this GD. Finally, we present the results obtained from networking these theories; these include a) an explanation, from the point of view of OSA, of some aspects that may be implicit in some of the mechanisms in APOS Theory, b) insights into the complementary nature of some constructs in these theories and c) the added value of this networking of theories as a means to find possible complementarities between the theories and to provide explanations to account for students' difficulties in constructing the derivative object.

2 Two different theoretical perspectives

In what follows we summarize some aspects of the two theories involved in this study.

2.1 OSA Approach

The OSA considers that in order to describe the mathematical activity from an institutional and personal point of view, it is essential to have in mind the objects involved in such activity and the semiotic relations between them (Font, Godino & Gallardo 2013; Rondero & Font, 2015).

Mathematical activity is modelled in terms of practices, configuration of primary objects and processes that are activated by practices. A mathematical practice is conceived in this theory as a sequence of actions, regulated by institutionally established rules, oriented towards a goal (usually solving a problem). In the OSA ontology, the term 'object' is used in a broad sense to refer to any entity which is, in some way, involved in mathematical practice and can be identified as a unit. E. g., when carrying out and evaluating a problem solving practice, we can identify the use of different languages (verbal, graphic, symbolic,…). These languages are the ostensive part of a series of definitions, propositions and procedures that are involved in argumentation and justification of the problem solution. Problems, languages, definitions, propositions, procedures and arguments are considered as objects, specifically as the six mathematical primary objects. Taken together they form configurations of primary objects. The term configuration is used to designate a heterogeneous set or system of objects that are related to each other. Any configuration of objects can be seen both from a personal and from an institutional perspective, which leads to the distinction of cognitive (personal) and epistemic (institutional) configurations of primary objects.

Depending on the language game in which they are involved the primary mathematical objects that form part of mathematical practices "may be considered in terms of how they participate, and the different ways of doing so may be grouped into dual facets or dimensions" (Font et al. 2013, p. 111). These are: Personal/institutional (see previous paragraph), unitary/systemic, expression/content, ostensive/non-ostensive and extensive/intensive.

If the perspective process-product is applied to the primary objects of configurations, a representation (language in OSA) can be considered as the result of a representation process, an argument is the result of a process of argumentation, a procedure relates to the process of automation, etc. Therefore, in order to analyze mathematical activity, besides practices and configurations of primary objects, OSA takes into account the processes derived from the application of process-product perspective to primary objects, together with the processes derived from the five dualities (personalisation-institutionalisation; synthesis-analysis; representation-signification; materialisation-idealisation; generalisation-particularisation). Other important processes taken into account are visualization and modelling, among others. E. g., to distinguish practice, primary objects and processes, consider the mathematical activity involved in the solution of the task: Calculate the derivative of the function $f(x)=\frac{x^2-3x+2}{3x-4}$. The student performs a sequence of actions (practice), such as reading the statement and calculating the derivative by means of the rule for the derivative of the quotient of functions: $f'(x)=\frac{(2x-3)\cdot(3x-4)-3(x^2-3x+2)}{(3x-4)^2}$ which is a procedure, considered as a primary object in OSA. When the student solves similar exercises, he or she engages in a process of automation.

2.2 APOS Theory

APOS (Arnon et al. 2014) is an acronym that stands for the types of mental structures (Action, Process, Object, and Schema) that students construct in the development of their understanding of mathematical concepts. In APOS Theory, a learner, usually referred to as a student, performs actions on previously constructed mental objects when involved in mathematical practice. An action is any transformation of objects according to an explicit algorithm, and is seen as being at least somewhat externally driven. As the student repeats the action or actions and reflects on them, they may be interiorized (by reflective abstraction) into a mental process. Reflective abstraction, as defined by Piaget, consists of drawing proprieties from mental or physical actions. It involves consciousness of the actions and the projection of the abstracted actions to a higher plane of thought (Arnon et al. 2014). An important characteristic of a process is that the individual is able to describe, or reflect upon, the steps of the transformation wholly in her/his mind without actually performing those steps. Processes can be reversed and coordinated with other processes to construct new processes. When students become aware of a process as a totality and are able to apply a new action on that totality, the process is encapsulated into a mental object. When necessary, a student may de-encapsulate an object back into its underlying process or think of the transformation in terms of actions.

Interiorization, coordination and encapsulation are the mechanisms involved in the construction of these mental structures. According to APOS, learning a mathematical concept involves applying a transformation to already constructed objects and through these transformations new mental or cognitive objects are constructed.

The student can construct relations between actions, processes, objects and schemas and construct a schema for a mathematical concept. A schema is defined in APOS Theory as an individual's collection of actions, processes, objects and other schema linked consciously or unconsciously in a coherent framework in the individual's mind. Individuals use their schemas to solve what they consider as related problems. A schema can be thematized into an object so that actions can be applied to it. Thematization is the mechanism involved in constructing an object from a schema.

APOS Theory includes, as part of its application to research and teaching, a general model describing a possible way to construct the concept or topic of interest. This hypothetical model is called the Genetic Decomposition (GD) which includes a theoretical analysis of the actions, processes, objects, and schemas that a learner may construct in order to learn a given/specific mathematical concept. It is used to design and implement research and instructional treatments. Research with APOS Theory follows a cycle with the following phases: Theoretical analysis and design of a GD, research to obtain a first validation of the GD, instructional design, instruction, research on instruction to validate the GD and the instructional design. This cycle can be repeated to refine the GD until it is considered to be validated experimentally.

3 Methodology

The authors of this paper consist of two researchers experienced in using APOS in their research and two researchers who have a substantial experience in using OSA. Independently of the fact that some authors use mainly one of the theories in their research, all of them have a deep knowledge of both theories. This fact is important to start the “networking” between the two theories.

The strategies used for connecting theoretical approaches were: 1) Understanding each theory and making own theories understandable; 2) comparing and contrasting and 3) making a coordination. We used the following methodology to carry out these networking strategies:

First we determined the way in which both theories model mathematical activity and how the nature of mathematical objects is conceived in each of them, explicitly or implicitly, to compare and contrast them and to outline a first and general idea of the possibility to coordinate them. This is an important starting point since several authors consider that determining the aspects characterizing a theory is necessary to compare and classify them (Ernest 1994; Radford 2008). Those essential aspects include principles, methods, and paradigmatic research questions (Radford 2008), and according to Ernest (1994), the principles characterizing each theory determine implicitly or explicitly, a position on the nature of mathematical objects.

Second we applied one of the basic principles of "networking of theories" (Bikner-Ahsbahs & Prediger 2010): Try to maintain the effort to coordinate the theories as concrete as possible. In accordance with this principle, a specific mathematical object (the derivative) was selected as context for reflection and analysis. The selection of this particular concept was a joint decision of all the authors upon consideration that there are relevant research studies dealing with the derivative using each theory, OSA (Font & Contreras 2008) or APOS (Asiala, Cottrill, Dubinsky & Schwingendorf 1997), as theoretical framework.

Third we decided to take the GD, a key theoretical notion of APOS, as the context for reflection. GD is a hypothetical model that describes the mental constructions related to the learning of a concept. As such, the GD includes those mechanisms that are directly related to the emergence of objects in APOS (encapsulation and thematization). With these ideas in mind a GD for the concept of derivative was designed.

Fourth we used OSA as a hand lens. The constructions of objects in the designed GD was then analyzed in order to examine the different constructions involved, in particular, those related to the construction of objects in APOS. Results of this analysis were discussed and negotiated by the team of researchers. The goal of this analysis was to gain a better understanding of the encapsulation and thematization mechanisms from APOS.

Fifth we compared similarities between some notions of APOS Theory −those used in the GD for the derivative, see step 3− and some of the concepts resulting from the analysis conducted with OSA −those considered part of the fundamental principles of this theory in Drijvers, Godino, Font and Trouche (2013)−, see step 4. For reasons of space, in this paper we limit our attention to the comparison of some principles of the two theories and do not include results regarding the comparison of methods and paradigmatic research questions, which are left for future research.

We finally discussed what we consider new knowledge gained in this research process. We describe in what follows the obtained results.

4 The nature of mathematical objects in both theories

The first step in the "networking of theories" consisted in a deep reflection on the notion of object, particularly on its nature, in both theories to respond the question: What is the nature of mathematical objects in both theories? In OSA the duality institutional-personal is considered a very important notion. In it, e.g. a distinction between epistemic configurations (EC) of primary objects (the institutional point of view) and cognitive configurations (CG) of primary objects (individual point of view) is made. In APOS the institutional point of view is taken into account in two different ways: 1) Mathematical objects are considered to be the result of the work of mathematicians who through their interaction in the mathematical community make their mental constructions explicit. 2) APOS Theory includes a didactical component –the ACE teaching cycle (Activities; Classroom Discussion; and Exercises)– which consists in the use of activities in a collaborative environment which gives learners opportunities to perform mathematical actions and reflect upon them to construct other internal structures, among which are mental objects. The activity session is followed by a group discussion session where new opportunities of reflection and formalization are offered through interaction with the teacher, and where homework exercises are proposed. The ACE cycle can be considered the institutional dimension of the APOS Theory. Constructed mental structures are considered in APOS both mental and mathematical since there is a dialectical relationship between them, which means that the mathematical object studied and the object constructed cannot be separated (Dubinsky 1997).

OSA's position is clearly a conventionalist constructivism (Font et al. 2013) inspired by Wittgenstein. These authors formulate the question: What is the nature of primary objects? They respond that whereas problems, linguistic elements and arguments do not change their appearance; procedures, definitions and propositions do. According to these authors, procedures are clearly rules, even though in many cases they are formulated as propositions (e. g., the quotient rule for the derivative of functions is stated in the form of a proposition), or even as definitions (e. g., if we define the perpendicular bisector of a segment as the perpendicular that passes through the mid-point, then we are implicitly stating a construction procedure). They also argue that definitions and propositions are particularly problematic as they seem to refer to mathematical objects which exist in some form or another.

In line with Wittgenstein's philosophy of mathematics, in the ontology proposed by OSA (Font et al. 2013) definitions and propositions are regarded as 'grammatical' rules. From Wittgenstein point of view, mathematical statements are rules governing the use of certain types of signs, since that is precisely how they are used, as rules. They do not describe properties of mathematical objects with any kind of existence independent of the people who wish to know about them or the language through which they are known, even if this may appear to be the case.

To explain the process from which a specific mathematical object emerges from practice, in OSA (Font et al. 2013), at least two levels need to be considered. In a first level, representations (language), definitions, propositions, procedures, problems and arguments (primary objects) emerge. Afterwards, in the second level, one single object emerges (e. g., the function object). Such object can (1) be associated with different representations, (2) has several equivalent definitions and (3) has properties, etc. Although in OSA a conventionalist point of view about the nature of the primary objects is assumed, it is not ignored that, in the teaching process, implicitly a descriptive/ realistic point of view of mathematics is suggested. This second level of objects explains how a realistic point of view is generated (e. g. only a "something" may have properties).

The emergence of mathematical objects in the APOS Theory is the result of individual's constructions when they act and reflect on mathematical tasks. The construction of mental structures related to mathematical objects results from the interaction and reflection on actions on mathematical objects. The construction of mental structures starts by performing and reflecting on actions on previously constructed objects. Through the mechanism of reflective abstraction, those actions can be interiorized into processes and when it is necessary to apply an action on a process that is considered as a totality, it is encapsulated into an object. The construction of relations between different structures related to mathematical objects leads to the construction of a schema. A schema is a dynamical structure. It is formed by different actions, processes, objects and other schemas that an individual has constructed in his experience of using and reflecting on mathematical activity which can be brought to bear in particular instances of her/his mathematical activity. Although a schema is continuously changing, as an individual faces different problems during her/his mathematical activity, it can become a mental object when actions are applied to it. E. g., when actions need to be applied to the function schema to verify if a set of functions constitute a vector space, it is thematized into an object. The mechanism involved in this transformation is called thematization.

APOS is formulated as a theory on Piaget’s epistemology, so it can be considered that mathematical objects are constructed as the result of the development of specific mental structures while individuals reflect on mathematical activity (e. g., in this theory it is considered that without abstractive reflection there is no possibility to construct objects). This possibility of development is shared by all those individuals who have experienced a similar biological and social development. This inter-subjectivity of the constructions can explain why mathematics is objective in the social world.

The position of both theories on the relation between mathematics and the external world is constructivist. In both of them mathematics are not considered a description of the external world but a human construction. However, each of them has a different point of view about the nature of the constructions. The institutional dimension is taken into account explicitly through specific constructs in OSA. In APOS this dimension is taken into account implicitly in the teaching cycle which is an important component of the theory. This constitutes an obvious difference between these two theories.

Thus, both theories consider the constructive nature of mathematics and take the institutional component into account. In both of them the mathematical activity of individuals plays a central role and both use notions involved in their description that show similarities (e.g. action, process or object). They also share a constructivist position in relation to the nature of mathematics. These similarities led authors to conclude that there are no intrinsic contradictions between the two theories, and that possible connections between them could be expected through their comparison. In order to be able to reflect on the commonalities and differences between APOS and OSA, we analyzed the different constructions described in a GD involved in learning a mathematical object. We then reflected on those constructions from OSA’s point of view.

5 A GD of derivative as a context for reflexion

The derivative was selected as the object of analysis. We reviewed as a starting point a GD, developed and validated experimentally by Asiala et al. (1997). Since recently, searching for clarity, a tendency to describe in more detail the different elements of a GD has emerged, we decided to analyze a recently developed GD for the same concept (Badillo 2003). After the analysis and comparison of both GDs we decided to design a new one by taking into account those constructions we considered important appearing in both GDs, and adding new constructions that were not considered explicitly in any of them which we considered important. This option is coherent with APOS Theory methodology since a GD for a concept is not supposed to be unique, and it is possible to refine a GD as a result of experimental validation, research results about a concept's learning and/or a triangulated decision by experts. In what follows we present a refined GD for derivative.

5.1 A GD for the derivative

The prerequisite concepts needed to construct a derivative schema are graphical and analytical representations of points and lines as objects, and graphical and analytical representations of functions as processes.

GD of the derivative

1a. The actions of (a) Connecting two points on a curve defined by a function in order to obtain a chord, that is, a piece of a secant line to the curve, passing through those two points, and (b) calculating the slope of the secant line passing through those points.

1b. Action of calculating the average rate of variation between a given point and a point that is "close" to it $\left(m = \overline{V} = TMV = \frac{f(b) - f(a)}{b - a} \right)$

2a. Interiorizing the actions in part *1a* into a process by reflecting upon the results of their repetition when the two points considered in the construction of the secant get closer together.

2b. Interiorizing the actions in part *1b* into a process of calculation of the average rate of variation, when the difference of the x-components from a general point to a given one gets smaller and smaller (b→a).

3a. Encapsulation of the process in part *2a* into (a) the object tangent line as the result of the limit of the position of the secants, and (b) the object slope of the tangent line to a point on the graph of a function as the result of the limit of the slopes of the secants.

3b. Encapsulation of the process in part *2b*, into the object instantaneous rate of variation of the dependent variable in relation to the independent variable $\left(\lim_{b \to a} \frac{f(b) - f(a)}{b - a} = f'(a) \right)$.

3c. Coordination of the processes in parts *2a* and *2b* into a new process where instantaneous rate of variation at a point is considered the same object independently of the representation used.

3d. Interiorizing the actions in part *1b* into a process where it is possible to consider the graphical construction of the average rate of change in terms of the quotient of vertical and horizontal increments when these increments are made smaller and smaller (*h→0*).

3e. Interiorizing the actions in part *1b* into an analytical process that involves the construction of the average rate of variation in terms of the quotients of increments of the dependent and independent variables when these increments are made smaller and smaller (*h→0*).

3f. Coordination of processes described in parts *3d* and *3e* into a new process where it is possible to consider both processes as the same independently of the representation used.

3g. Encapsulation of the process described in part *3f* into the object instantaneous rate of variation in any point of a function considered as the limit of the quotients $\left[\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \right]$.

4a. Actions of (a) connecting two different but close points on a graph, relating two co-varying magnitudes by means of a line segment and calculating its slope or (b) using two consecutive points of a table relating two co-varying magnitudes to calculate the average rate of variation.

4b. Interiorizing the actions described in part *4a* into processes to obtain an average rate of change as a reasonable approximation of the instantaneous rate of change at a given point.

4c. Encapsulation of the processes in part *4b* into objects that are equivalent to the instantaneous rate of change at a point, in a numerical or graphical context $\left(\frac{\Delta y}{\Delta x}\right)$. (In the case of velocity this would mean $\left(\frac{\Delta d}{\Delta t}=\bar{v}\right)$).

4d. Coordination of the processes described in part *4b* and part *3d* into a new process in which instantaneous rate of change can be obtained from a graph or from data sets by a linear approximation.

4e. Encapsulation of the process in part *4d* into the object instantaneous rate of change at a point, as the limit of the average rates of change at a point $\left(\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\right)$.

5. Establishing relations between the objects resulting from encapsulations in parts *3b*, *3g* and *4e* to construct a schema for the derivative at a point.

6. Thematization of this schema into the derivative object of a function at a point $\left(m_{\tan}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{b\to a}\frac{f(b)-f(a)}{b-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)\right)$.

7. Interiorizing the actions of calculating the derivative at a point into a process to calculate the derivative at any point of a function.

8. Encapsulation of the process described in part *7* into the derivative object as a function.

8a. Actions on the derivative as an object to find the derivative of functions that result from operations with other functions.

8b. Interiorization of actions in part *8a* into processes involved in the calculation of the derivative of functions obtained from operations, that is, obtaining the derivative rules.

8c. Encapsulation of the processes in *8b* into objects related with derivative rules. E. g., actions are applied to the product rule for the derivative in integration by parts.

9. Establishing relations among processes or objects constructed in *6* and *8* into a derivative schema that makes finding the properties of any function by means of its derivative and the properties of the derivative of a function possible (monotony, concavity, description and construction of graphs, optimization, etc.).

10. Thematization of the derivative schema when actions need to be applied to the schema to consider, e. g. the integral of a derivative function in any representation.

As mentioned before, the GD is a model of constructions needed to learn mathematical objects. It is not supposed that the constructions described are constructed in a specific linear way by students. The numbers we include in this GD are intended to facilitate the reading and the discussion that follows and not necessarily to describe specific steps in the learning of the derivative.

As can be observed, in this GD there are two different mechanisms associated to the construction of objects in APOS: Encapsulation (3a, 3b, 3g, 4c, 4e, 8 and 8c) and thematization (6 and 10). The resulting objects are also different in nature. Reflection about these different mechanisms and the differences in the emerging objects may help to understand these mechanisms better. This will be described in the following sections.

In APOS processes are considered to be encapsulated into objects when they can be considered as a whole in order to apply actions on them; and that schemas are thematized into objects when it is possible to apply actions on them. The mechanism involved in this later transition is thematization. A schema has been thematized when there is evidence that the student can think of it as a total entity and perform actions on it, e. g. when operating with it or comparing it with another schema, or when it can be decomposed to recover its components, and/or to make the necessary actions and processes to reconstruct it when conditions of the problem situation are changed (Cooley, Trigueros & Baker 2007). The derivative schema would be thematized when, e. g., the student does the action of calculating the integral of a derivative function, or the action of constructing and analyzing a differential equation.

After reflecting on the construction of objects in APOS Theory through this GD, the authors of this study concluded that APOS' description of the construction of objects could be complemented by the problematization of the nature of the mathematical objects which emerge from the mechanisms of encapsulation and thematization. Since OSA approach entails a detailed description for the emergence of mathematical objects, it was decided that it could provide elements to better understand the complexity involved in the relation between the cognitive and the mathematical objects. Its use would thus allow the development of a local coordination between the two theories. This is illustrated in what follows, using the previously designed GD and focusing on the construction of objects from the perspective of OSA.

5.2 A look at the GD from the OSA

Using the GD for the derivative presented above, we analyze the emergence of objects from the mechanisms of encapsulation and thematization with the tools of OSA.

Encapsulation

In the GD, the word encapsulation appears for the first time in *3a* and *3b*. In *3b* an implicit definition of the mathematical object that results from the mechanism of encapsulation is included.

From the point of view of OSA, the transitions from actions to process and from process to object involve complexities that can be described by means of its theoretical tools. The student needs first to understand that the performed actions can be realized following a specific procedure (a rule that gives information about the order in which the actions must be performed to reach a certain goal). According to the OSA, a certain level of reification is produced in the transition from actions to processes: since the procedure can be treated as a unit, it can become a mathematical object. Then, in the GD, the student needs to consider a new object which results from the encapsulation of a process and, when this happens, he/she understands the meaning of the definition that provides information about this new object. From the OSA perspective, encapsulation of a process is thus a mechanism in which two different aspects can be considered: (1) different primary mathematical objects, and (2) the intervention of a dense weave of semiotic functions that relates actions and processes with procedures, and objects with definitions. This second aspect will now be briefly explained.

Meaning in OSA is basically conceived of in two ways. Firstly, meaning is conceived of through a semiotic function (Expression → Content), which is the correspondence, or dependency relationship between an antecedent (expression) and a consequent (content) established by a subject (or an institution) according to a rule, habit or criterion. Secondly, it can be understood in terms of usage. These two ways of understanding meaning complement each other since mathematical practices involve the activation of configurations of primary objects and processes that are related by means of semiotic functions. In this paper we will not detail the weave of semiotic functions that a student needs to activate in order to understand the meaning of the derivative at a point. This weave of semiotic functions can be obtained as adaptation of the set of semiotic functions (proposed in Font & Contreras 2008) that a student needs to activate in order to understand the derivative function, and can be schematically summarized as follows:

Generation of a succession of mean rates of variation that approach the limit *f'(a)* (procedure) → Class of $\left(\frac{f(b)-f(a)}{b-a}\right)$ as b→a (application of the unitary/systemic and extensive/intensive dualities → An object (a number) different from the previous class $\left(\lim_{b\to a}\frac{f(b)-f(a)}{b-a}=f'(a)\right)$ (definition) → the name instantaneous rate of variation (notational objet).

What is important here is to stress that when a process is encapsulated a double change in the nature of the object is produced. E.g. in *3b* one change occurs when a cognitive object is produced from the process, as emphasized by APOS Theory, and the other, signaled by OSA, is a change in the nature of the primary mathematical object, as it changes from a procedure to a definition. This double change can also be identified in encapsulations described in *3g* and *4e*. In contrast, according to OSA, in encapsulation described in *8c*, the change in the nature of the primary object, consists in a procedure becoming a property. In fact, it is significant that some texts call the property a rule (e. g. the chain rule). From OSA perspective, the change in the nature of a primary object produced in the course of encapsulation can be understood by the nature of definitions, procedures and, properties as rules, as described in the process of emergence of objects in section 4.

Thematization

In the GD we have presented above, we included the construction of objects resulting from thematization of schemas. In *6* the schema of derivative at a point is thematized. The different representations of the derivative are considered as the same object. Construction *10,* in contrast relates to thematization of the schema of derivative as a function. Although it may appear from the GD that these steps are trivial to construct, thematization is a slow process since many relations among components of the schema are constructed and transformed before the schema can be considered as a coherent totality and thematized.

According to OSA, the derivative object resulting from thematization must be situated in the second level of the emergence of objects. It has to do with the emergence of a global reference associated to different cognitive configurations developed by students (see, Fig. 1). As a result of classroom implementation of different epistemic configurations, students are able to work on mathematical practices related to the derivative, interpreted as limit, slope of the tangent line, or instantaneous velocity. These practices contribute to understanding that the derivative can be defined in different forms, that it can be represented in different ways, etc. The result is that it can be considered that there is an object, called derivative that plays the role of a global reference to all the configurations.

In OSA, the emergence of those objects that are considered as global reference of one or several configurations of primary objects is explained by the joint effect produced by the processes associated to the different dualities. The duality unitary-systemic makes it possible to consider a configuration, e. g. the EC limit in figure 1, as an object (derivative), or the set formed by the three configurations in figure 1, as an object (derivative). The other dualities are important in order to point out the properties attributed to that object. The expression-content duality makes it possible to duplicate the object considering the representation and the represented object as different objects (this leads to thinking that the different configurations are representations or partial descriptions of the derivative object). The ostensive-non ostensive duality leads to considering that the represented object is an ideal object different from its material representations, the extensive-intensive duality leads to considering, generally, that object as a general "something" and the duality personal-institutional leads to considering that object as inter-subjective. The combination of these dualities produces the emergence of a global reference on which it is possible to perform certain actions.

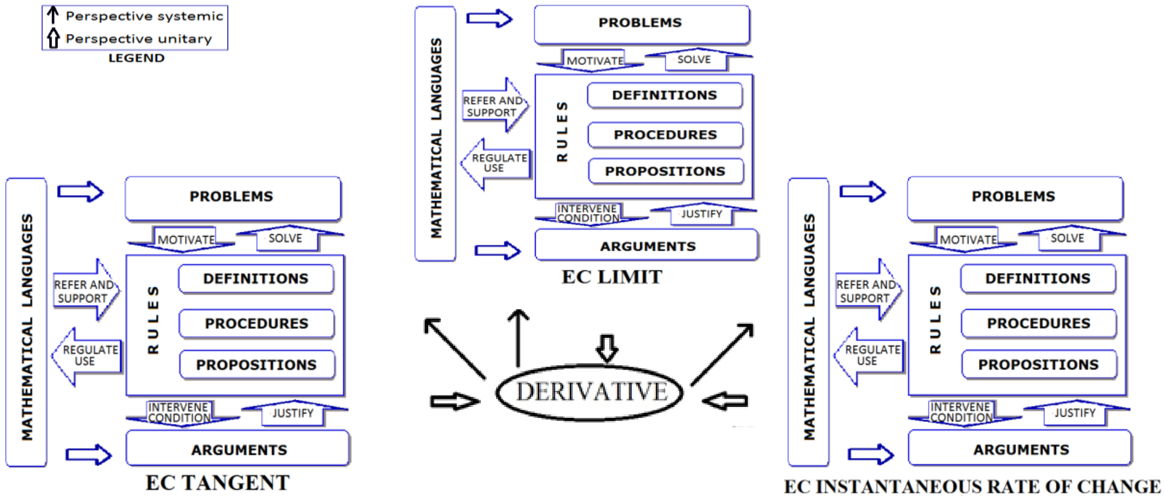

Figure 1. The derivative object as a global reference.

It is possible then, if we consider the point of view of OSA, to relate the thematization mechanism with different dualities involved in the emergence of second order objects. Furthermore, OSA's description clarifies why the thematization process is so slow, as described in APOS Theory (Cooley et al. 2007): The students have to establish a lot of connections between all the parts that form a new entity (as a global reference object). This complexity shapes the frame for disclosing students' difficulties with the derivative concept.

6 The link between these frameworks

In networking these theories, we considered an analysis in terms of comparing and contrasting some of the principles of both theories. In what follows we describe specific results obtained as a result of our reflection.

Action versus practice

The notions of practice (in OSA) and action, or set of actions (in APOS) are complementary. The main difference consists in the fact that actions in APOS are mental transformations while practice in OSA is a set of actions governed by rules (known by the community). In APOS actions are any transformation or sequence of transformations of cognitive objects according to an explicit algorithm or that is seen as being externally driven. E. g., given a formula for a function of two variables and a point, an individual is able to calculate the value of the function at that point. When actions are interiorized into a process, they are no longer externally driven and their result can be reached without having to perform each of them. In the OSA a practice is understood as a sequence of mathematical actions governed by rules. The set of rules governing the practice is categorized into different types of primary objects, which form a cognitive/epistemic configuration.

Process versus procedure

In APOS Theory, as actions are repeated and the individual reflects upon them, they may be interiorized into a process. An important characteristic of a process is that the individual is able to describe, or reflect upon, the steps of the transformation wholly in her/his mind without actually performing those steps. Additionally, once a mental process is constructed, it is possible for an individual to reverse it and construct a new process (a reversal of the original process). When the individual is able to describe the rules needed to find the partial derivatives of a given two-variable function, e. g., he/she has interiorized the actions of finding the partial derivatives into a process. If the individual is able to think of the process of finding a function with those particular partial derivatives, then he/she shows to have constructed a new process (e. g., integration with respect to a particular variable).

As mentioned before, in OSA a practice is understood as a sequence of mathematical actions governed by rules that can be categorized as different types of primary objects. One of these objects is the procedure, which is understood as a rule indicating the steps to carry out the practice, or a part of the practice. A procedure can also be considered complementary to an APOS process when a student shows to have interiorized cognitive actions and is aware of the result of the process.

Encapsulation versus primary object

The result of encapsulation of a process in APOS is a mental object. In other words, this can be stated by saying that the result of encapsulation is “something” upon which it is possible to perform new actions. From the point of view of OSA, encapsulation produces a double change in nature. On the one hand, from process to object (a primary object according to OSA), as described by APOS Theory. On the other hand, it produces a change in the nature of the mathematical object associated to the cognitive object, since there is often a change from a procedure to a definition, as we have exemplified previously while discussing the GD of the derivative. Knowing that this change occurs allows giving a better explanation to students’ difficulties with the concept of derivative, or with other concepts that may be studied in the future.

Cognitive configuration versus schema

The notions of cognitive configuration and schema can also be considered as complementary. From our analysis, the schema can be formed by different structures (Actions, Processes, Objects, Schemas) related among them, and the notion of cognitive configuration includes a typology of mathematical objects and processes. With the notion of cognitive configuration a structure can be proposed for the schema when it is used in a given situation. The schema can be considered as a cognitive description of the subject’s mental organization that is used to respond in similar problem situations. But when we analyze the implementation of a schema in context, we can recognize a structure that is systematically described by OSA in terms of cognitive configurations. Therefore, a cognitive configuration may disclose the complexity of a specific task related to a schema when it is presented to students. Hence, schema and cognitive configuration are distinct but complementary notions. A cognitive configuration is linked to a particular task, whereas a schema concerns types of tasks.

Thematization and the second level of emergence of an object

According to the APOS Theory, an object emerges from thematization of a schema. When a schema has developed, as a result of a change in the nature of the relations among its component structures, into a coherent structure that can be considered as a whole, it is thematized into an object and actions can be performed on it. When the mathematical object emerging from thematization is considered from the point of view of OSA, thematization can be related to the second level of emergence of objects. The mathematical object that can be related to a schema can be considered as a global reference associated to different cognitive configurations or systems of cognitive configurations. Again, APOS and OSA can complement each other. While APOS describes in detail the components and relations in a schema that is useful in working with a set of mathematical tasks related to a concept, in OSA each task is analyzed through a cognitive configuration which indicates specific differences that distinguish those tasks and may explain specificities that may lay behind students' difficulties.

7 Final considerations

In this paper a networking of two theories, APOS and OSA, was presented. A GD was designed and used to carry out a comparison of these theories focusing on the notion of object and on its nature. This comparison, taking into account their structures, permitted to gain a deeper understanding of their commonalities and differences, and also to make a local coordination.

This comparison put forward interesting commonalities in both theories: Both of them position themselves as constructivist theories regarding the relation between mathematics and the real world. Although each of them has a different position in terms of the nature of the constructions, in both theories mathematical objects are not considered to exist independently of the people who learn about them, and both use similar notions when describing mathematical activity. Based on these similarities, it can be concluded that there are no intrinsic contradictions between them.

Particular results obtained from the discussion about the notion of object in both theories were possible by the detailed description of constructions included in the GD. The use of OSA point of view to analyze it led to: (1) The recognition of a double change in nature of the emergent object from encapsulation (sometimes from procedure to definition and others from procedure to property) and (2) the relation between thematization and the dualities involved in the emergence of second order objects.

Networking of these theories was useful in underlying local coordination that provides tools to explain the difficulties associated with the learning of specific concepts. This was shown in this paper in particular for the derivative object. Encapsulation and thematization are considered in the APOS Theory as the mechanisms involved in the construction of objects. Results of research, using the APOS Theory, show that students have difficulties in the construction of both kinds of objects (Cooley et al. 2007). Taking into consideration OSA's point of view, some aspects that may be implicit in the encapsulation and thematization mechanisms, are shown which may help to better understanding of the complexity associated to them and to the emergent objects (e. g., when the derivative is built as a cognitive object in a specific way, but a problem implies to use the derivative in a different way, then students would have difficulties in solving it). These theories complement each other showing a direction of research on student difficulties with the derivative that may provide more insights than we had before. Moreover, the information from our research study can contribute to the knowledge provided by other studies focusing on the topic of teachers learning to notice students' mathematical thinking about derivative, which have used the APOS Theory (e. g., Sánchez-Matamoros, Fernández & Llinares 2014).

Some interesting links were found between both theories: The notions of practice and action, on one hand, and process and procedure, on the other hand, are complementary. Schema and cognitive configuration can also be considered as complementary, since it may be possible to use the notion of cognitive configuration to propose a structure for a schema used in particular situations.

The networking of APOS and OSA theories conducted in this research has enabled us to compare them and to disclose the complementary nature of some of their constructs. We have found, that APOS and OSA, as lenses for didactical analysis, have some commonalities and, more importantly, that they do not contradict each other. We can say that certain connectivity between them is possible, even though only the notion of object was studied.

Results obtained may also help when didactical interventions are designed with each of these theories, since the complexities involved in the construction of objects and the interpretation of the mechanism involved in the construction of cognitive objects have been clarified and specified. More research is needed to study in detail this possibility and to state the kind of results that can be obtained.

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